Lecture 37

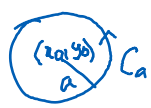
Tuesday, June 25, 2024 8:32 AM

Ex $F(x,y) = (xy, x+y)$

$G(x,y,z) = (z, yx, z^2+x)$

Find curl and divergence of F and G.

* Geometric meaning of curl and divergence:



$$\int_{C_a} F.dr = \text{work done by } F \text{ along } C_a$$
$$= \text{circulation of } F \text{ along } C_a$$

By Green's theorem:

$$\int_{C_a} F.dr = \iint_{D_a} (Q_x - P_y)\,dA \approx \left(Q_x(x_0,y_0) - P_y(x_0,y_0)\right)\text{area}(D_a)$$

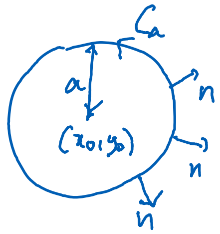
$$\rightsquigarrow \quad \frac{1}{\text{area}(D_a)} \int_{C_a} F.dr \approx Q_x(x_0,y_0) - P_y(x_0,y_0)$$

In fact,

$$\lim_{a\to 0} \frac{1}{\text{area}(D_a)} \int_{C_a} F.dr = Q_x(x_0,y_0) - P_y(x_0,y_0) = \text{curl}\,F(x_0,y_0)$$

If $\text{curl}\,F(x_0,y_0) > 0$ then $\int_{C_a} F.dr > 0$ for small a, so F spins counterclockwise around (x_0,y_0).

If $\text{curl}\,F(x_0,y_0) < 0$ then " < 0 " " " clockwise.



$$\int_{C_a} F \cdot dn = \text{flux of } F \text{ accross } C_a$$

$$\lim_{a \to 0} \frac{1}{\text{area}(D_a)} \int_{C_a} F \cdot dn = \text{div} F(x_0, y_0)$$

If $\text{div} F(x_0, y_0) > 0$ then (x_0, y_0) is a source point.

If $\text{div} F(x_0, y_0) < 0$ then (x_0, y_0) is a sink point.

<u>Surface parametrization</u>

Curve	Surface
$r(t) = (x(t), y(t))$ $r(t) = (x(t), y(t), z(t))$	$r(u,v) = (x(u,v), y(u,v), z(u,v))$

Ex the surface $z = f(x,y)$ can be parametrized by

$$\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$$

Ex the plane $x + y + z = 1$ can be parametrized by

$$\begin{cases} x = u \\ y = v \\ z = 1 - u - v \end{cases} \quad \text{or} \quad \begin{cases} x = 1 - u - v \\ y = u \\ z = v \end{cases}$$

Ex the surface $r(u,v) = (\underbrace{u\cos v}_{x}, \underbrace{u\sin v}_{y}, \underbrace{v}_{z})$ is a helicoid.

On Mathematica: ParametricPlot3D[{u Cos[v], u Sin[v], v}, {u,0,10}, {v,0,10}]

The mesh on the surface are the curves $u = \text{const}$ and $v = \text{const}$.