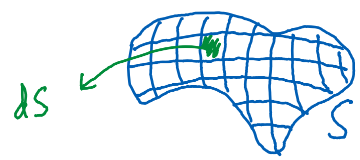
Lecture 38

Wednesday, June 26, 2024 9:26 AM

* Surface integral: Imagine a metal plate of the shape of a surface S. It has uneven mass distribution. The mass density is



$$f(x,y,z) \quad \left(\frac{\text{mass}}{\text{area}}\right)$$

Area of a cell is dS. Note that the letters "S" in the surface S and in dS are not the same. Some textbooks use $d\sigma$ or $d\Sigma$ instead of dS. Or we could have named the surface by something else other than S. However, because this abuse of notation is so common (at least in Multivariable Calculus), we will also write dS for cell area and S for the name of the surface.

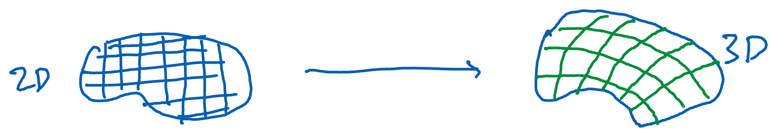
S is parametrized by $r(u,v) = (x(u,v), y(u,v), z(u,v))$.

$$dS = |r_u \times r_v| \, du\,dv$$

The stretching factor $|r_u \times r_v|$ is analogous to the Jacobian.

Stretching factor $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

Stretching factor $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right|$

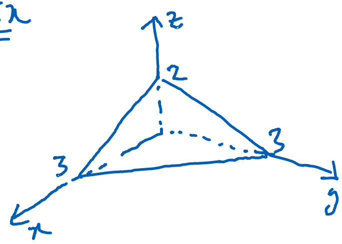


(u,v) $\qquad\qquad$ (x,y,z)

Stretching factor $|r_u \times r_v|$

Total mass $= \iint_S f(x,y,z)\,dS$

Ex



Compute $\iint_S x^2\,dS$ where S is the triangle with vertices at $(3,0,0)$, $(0,3,0)$, $(0,0,2)$.

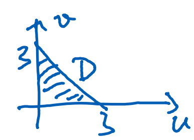
Equation of the surface is

$$\frac{x}{3}+\frac{y}{3}+\frac{z}{2}=1$$

Parametrization:

$$\begin{cases} x=u \\ y=v \\ z=2\left(1-\frac{u}{3}-\frac{v}{3}\right) \end{cases}$$

$(u,v)\in D$,



$$r(u,v)=\left(u,v,2-\frac{2u}{3}-\frac{2v}{3}\right)$$

$$\left.\begin{aligned} r_u &= \left(1,0,-\tfrac{2}{3}\right) \\ r_v &= \left(0,1,-\tfrac{2}{3}\right) \end{aligned}\right\} \quad r_u \times r_v = \left(\frac{2}{3},\frac{2}{3},1\right)$$

$$|r_u \times r_v| = \sqrt{\frac{4}{9}+\frac{4}{9}+1} = \frac{\sqrt{17}}{3}$$

$$\iint_S x^2\,dS = \iint_D u^2\,|r_u\times r_v|\,du\,dv = \frac{\sqrt{17}}{3}\iint_D u^2\,du\,dv = \frac{\sqrt{17}}{3}\int_0^3\int_0^{3-u} u^2\,dv\,du = \frac{9\sqrt{17}}{4}$$