Math 301

Sets : collection of items

* Use capital letter for sets, lowercase for elements.

An item can be inside or outside of a set.

A set X is defined to be the set of all items that don't belong to itself.

$X = \{ a \mid a \notin a \}$ $X \in X$? $X \notin X$

Russell's Paradox

Descriptions of a set

- list all elements

ex.

$X = \{1, 2, 4, 7\}$
$= \{1, 1, 1, 2, 4, 7\}$
$= \{2, 1, 7, 4\}$

order and repetition doesn't matter

- list with ellipsis :

ex.

$X = \{1, 3, 5, 7, 9, \ldots, 101\}$

not good:

$X = \{2, 4, \ldots, 64\}$

- use set builder notation:

such that

$X = \{\text{natural number } a \mid a \text{ is divisible by } 3\}$

s.t.

: (colon)

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

0 is not natural number

Number System

- natural numbers $\mathbb{N}$ (blackboard font)
- integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
- rational numbers $\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$
- real numbers $\mathbb{R}$ ________

  $\mathbb{R} = \{\lim_{n \to \infty} a_n \mid (a_n) \text{ is a Cauchy sequence}, a_n \in \mathbb{Q}\}$
- complex numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, $i^2 = -1$
- quaternion $\mathbb{H} = \{ai + bj + ck + d \mid i^2 = j^2 = k^2 = ijk = -1\}$

1.4 Exercises

(a) $A_1 = \{\cancel{1,2,3,4,5,\ldots}\} = \{1\}$

(b) $A_2 = \{\cancel{\pm 2, \pm 3, \pm 4} = \cancel{\{\pm 1\}} = \{0, \pm 1\}$

(c) $A_3 = \{\cancel{\ldots}\}$

$x = 3k = \frac{216}{m}$ X is a multiple of 3 and a divisor of 216

$x = km = \frac{216}{3}$

$x = km = 72$

(d) $\frac{x+2}{5} \in \mathbb{Z}$

5/2/2025

Math 301

(1) Find all divisors (positive) of 18.

(2) Find all real numbers x such that $x^2-9x+20=0$

(1) $\{1,2,3,6,9,18\}$

(2) $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \dfrac{9 \pm \sqrt{81-4(20)}}{2}$

$x = \dfrac{9 \pm \sqrt{1}}{2}$

$= \dfrac{9+1}{2}, \dfrac{9-1}{2} = \dfrac{10}{2}, \dfrac{8}{2}$

~~$x = \{5,4\}$~~

$x = \{4,5\}$

Need a full sentence!

① All positive divisors of 18 are 1, 2, 3, 6, 9, and 18.

② The lefthand side of the equation can be factored as $(x-4)(x-5)$. The equation can be written as

$(x-4)(x-5)=0$. Thus, one of the two factors must be zero. That is either $x-4=0$ or $x-5=0$.

Therefore, $x=4$ or $x=5$.

Chapter 2: Logic

Statement is a sentence that you can assign a value of true or false.

→ Proposition

Ex: 2 is an even number. - True statement

1 is an even number. - False statement

1 is a better number than 2. - Not statement

$x = 1$

↳ is an open sentence.

In other words, it would be a statement if you know the value of x.

Negation of a statement P is $\sim P$. (not P)
↑ tilde.

P and Q : $P \wedge Q$ conjunction statement

P or Q : $P \vee Q$ disjunction statement

Truth Table

P	$\sim P$	
T	F	Truth
F	T	False

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$\sim(P \wedge Q)$ vs $(\sim P) \vee (\sim Q)$

P	Q	$\sim(P \wedge Q)$	$(\sim P) \vee (\sim Q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Two statement are logically equivalent if they always have the same truth value.

$(P \wedge Q) \vee (\sim P \wedge \sim Q)$

P	Q	
T	T	T
F	T	F
T	F	F
F	F	T

1	2	3	4	5	6	
P	Q	$P \wedge Q$	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$	$3 \vee 6$
T	T	T	F	F	F	T
T	F	F	F	T	F	F
F	T	F	T	F	F	F
F	F	F	T	T	T	T

5.

(a) The sentence "8 is even and 5 is prime." can be written as

$(P \wedge Q)$, where P: "8 is even" and Q: "5 is prime."

P	Q	$P \wedge Q$
T	T	T

since both P and Q are true, the statement $(P \wedge Q)$ is true.

* If P then Q

$P \Rightarrow Q$

(b) The sentence "If n is a multiple of 4 and 6, then it is a multiple of 24" can be written as $(P \wedge Q) \Rightarrow R$, where P: n is a multiple of 4, Q: n is a multiple of 6, and R: n is a multiple of 24. Also, the truth value of this sentence depends on n, so it is an open sentence, not statement.

(c) If n is a not multiple of 10, then it is a multiple of 2 but is not a multiple of 5.

$(\sim P) \Rightarrow (Q \wedge (\sim R))$?

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Chapter 1

Problem 8 : Are the following sets equal?

X (a) $\mathbb{Z}$ and $\{a : a \in \mathbb{N}$ or $-a \in \mathbb{N}\}$ Natural number

The two sets, $\mathbb{Z}$ and $\{a : a \in N$ or $-a \in N\}$, are equal because $\mathbb{Z}$ is the set of all integers and the set $\{a : a \in N$ or $-a \in N\}$ contains all the whole number, both positive and negative.

✓ (b) $\{1, 2, 2, 3, 3, 3, 2, 2, 1\}$ and $\{1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3\}$

Since both of the sets contain the same elements, ~~regarding~~ regardless of ~~the~~ how many times the elements were repeated, the sets are still equal.

(c) $\{d : d$ is a day with 40 hours$\}$ and $\{w : w$ is a week with 6 days$\}$

* If you have a long set, give it a name.

Ex:

Let $A = \{a : a \in \mathbb{N}$ or $-a \in \mathbb{N}\}$

Conditional statement :

$P \Rightarrow Q$ "P implies Q"

"Q whenever P"

hypothesis / antecedent — conclusion / consequence

"If P then Q"

"P only if Q"

If $\underbrace{n \text{ is a prime number}}_{P}$ then $\underbrace{n^2 \text{ is a prime number}}_{Q}$.

$P \Rightarrow Q$ is false only when P is true and Q is false.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

If $1=2$ then $2=3$: $P \Rightarrow Q$

P: "$1=2$"

Q: "$2=3$"

~~Therefore, it is~~ $\boxed{\text{true statement.}}$

conditional statement $P \Rightarrow Q$.

$Q \Rightarrow P$: converse of the above statement.

$(\sim P) \Rightarrow (\sim Q)$: inverse of the above statement

$(\sim Q) \Rightarrow (\sim P)$: contrapositive of the above statement

$(P \Rightarrow Q)$ and $((\sim Q) \Rightarrow (\sim P))$ are logically equivalent.

$(Q \Rightarrow P)$ and $(\sim P \Rightarrow \sim Q)$ are logically equivalent.

$P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

P is equivalent to Q.

P if and only if Q.

P iff Q.

P is a necessary and sufficient condition for Q.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Chapter 2

Problem 6

(a) $(x \in R) \Rightarrow (x^2 \in R) \wedge (x^2 > 0)$

* Let $P: (x \in R)$ and $Q: (x^2 \in R) \wedge (x^2 > 0)$.

~~If~~ * Let $P = (x \in R)$ and $Q = (x^2 \in R) \wedge (x^2 > 0)$

If P

If x is an element of the set of all real numbers then x^2 is an element of the of all real numbers and x^2 is greater than 0.

If x is a real number then x^2 is a real number and x^2 is greater than 0.

(b) $4 \in \{2l : l \in N\}$

4 is a positive even number.

(c) $(x \in N) \Rightarrow \sim (x^2 = 0)$

If x is a natural number then x^2 is not equal to 0.

(d) $(x \in Z) \Rightarrow (x \in \{2l : l \in Z\}) \vee (x \in \{2k+1\} : k \in Z)$

5/7/2025

Math 301

* Practice writing definitions

Even

- An integer n is even if $n = 2k$ for some $k \in \mathbb{Z}$.

In a definition, "if" is the same as "if and only if"

<u>Ex: Definition for perfect square.</u>

n is a perfect square if $\sqrt{n} = k$ for some $k \in \mathbb{Z}$.

~~$n \in \mathbb{N}$ is a perfect square if $A =$~~

~~= x is a~~

For definition,

using square root ($\sqrt{\ }$) is not too friendly.

$z, y, x \rightarrow$ for real numbers

$n, m \rightarrow$ for natural, integers

$p \rightarrow$ prime numbers

* A number n is a perfect square if $n = a^2$, for some $a \in \mathbb{Z}$.

Exercise:

Write the definition of

1) a prime number

2) a is divisible by b

3) quotient and remainder of a division $a \div b$

where $a \in \mathbb{Z}$, $b \in \mathbb{N}$.

(1) A number p is a prime number if

- p has only 2 factors, 1 and itself
- p

(2) ~~a is divisible by~~ A number a is divisible by b

such that $a \in \mathbb{Z}$ if $\frac{a}{b}$ for some $b \in \mathbb{N}$.

~~such that~~ if $\frac{a}{b} \in \mathbb{Z}$.

(3)

Answer key:

(1) A prime number is a natural number greater than 1 which has only two divisors : 1 and itself.

(2) A number a is divisible by b if $\frac{a}{b} \in \mathbb{Z}$.

$(r \leq b) \in \mathbb{N}$

(3) A number $q \in \mathbb{Z}$ is a quotient and $r \in \mathbb{N}$ is a remainder of the division $\frac{a}{b}$ if $a = q(b) + r$.

$r \in \mathbb{N}$, where $r < b$, but

* The issue of this definition is that b hasn't mentioned yet.

A number $q \in \mathbb{Z}$ is a quotient and $r \in \mathbb{Z}$ is a remainder of $\frac{a}{b}$ if $a = q(b) + r$ and $0 \leq r < b$.

Most statements you are asked to prove are of the form "$P \Rightarrow Q$"

Trivial cases: P is false or Q is true.

In either cases, the implication $P \Rightarrow Q$ is true.

Ex: Prove that if $\underbrace{x^2 + 1 = 0 \text{ for some number } x \in \mathbb{R}}_{P}$ then $\underbrace{x^2 = 0}_{Q}$.

Direct Proof

- you assume the hypothesis P is true.
- Then you use a chain of logic to arrive at Q being true.

Ex: Prove that if $n \in \mathbb{Z}$ then $n^2 + 2n + 1$ is a perfect square.

① Assume $n \in \mathbb{Z}$

② we can factor $n^2 + 2n + 1$ as follows,

$$n^2 + 2n + 1 = (n+1)(n+1) = (n+1)^2.$$

③ Since $n \in \mathbb{Z}$, then $n+1 \in \mathbb{Z}$ as well.

④ Therefore $(n+1)^2$ is an integer, which by definition is a perfect square.

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Prove $P \Rightarrow Q$ by direct proof.

- Assume P is true.
- Use a chain of logics to imply Q is true.

Write " $a \equiv b \pmod{n}$ " $\longleftrightarrow$ (a is congruent to b in modulo n)

if $a-b$ is divisible by n.
$\hookrightarrow b-a$

Ex: $4 \equiv 1 \pmod{3}$
$5 \equiv 17 \pmod{2}$

Observation: $a \equiv b \pmod{n}$
if and only if a and b have
the same remainder in the division by n.

Ex: Prove that if $n \equiv 2 \pmod{5}$ then $n^2 \equiv 4 \pmod{5}$.

- Assume $n \equiv 2 \pmod{5}$ is true. We want to show that $n^2 \equiv 4 \pmod{5}$.

We have $n-2$ is divisible by 5. There is $k \in \mathbb{Z}$ such that
$n-2 = 5k$. [Alternatively, we can say "So, $n-2 = 5k$ for some $k \in \mathbb{Z}$."]

We want to show that $n^2 - 4$ is divisible by 5.
(That is to show $n^2 - 4 = 5k$ for some $k \in \mathbb{Z}$.) ← (*): This is wrong since k has already been used!

That is to show $n^2 - 4 = 5\ell$ for some $\ell \in \mathbb{Z}$.

We have $n = 2 + 5k$.
Squaring both sides gives $n^2 = (2+5k)^2 = 4 + 20k + 25k^2$.
So, $n^2 - 4 = 20k + 25k^2 = 5(4k + 5k^2)$.
We can choose / We have found $\ell = 4k + 5k^2$

Fact: for any $x \in \mathbb{R}$, we have $x^2 \geq 0$.

Fact: If $a \geq 0$ and ~~also~~ $b > 0$ then $a+b > 0$.

Fact: If $a > b$ and $c > 0$ then $ac > bc$.
If $a > b$ and $c < 0$ then $ac < bc$.

Fact: If $x > 0$ then $\frac{1}{x} > 0$.
If $x < 0$ then $\frac{1}{x} < 0$.

Fact: If $a \geq b$ then $a+c \geq b+c$.

Ex: Prove that $x \in \mathbb{R}$ then $x^2+1 \geq 2x$.

Assume that $x \in \mathbb{R}$ is true. We want to show that $x^2+1 \geq 2x$.

(~~We have for any $x \in \mathbb{R}$, we have $x^2 \geq 0$.~~)

We have for any $x \in \mathbb{R}$, we have $x^2 \geq 0$.

So we can write $x^2+1 \geq 2x$ as follows:

$$x^2+1-2x \geq 0.$$

[Remarks] This is equivalent to

$$(x-1)^2 \geq 0.$$

Since $x \in \mathbb{R}$, then $(x-1)^2$ is non negative.

There fore, it is greater than zero.

$$5(x+3) > 2x$$
$$5x+15 > 2x$$
$$5x-2x > -15$$
$$\frac{3x}{3} > \frac{-15}{3}$$
$$x > -5$$

5/12/2025

Ex: Show that if $|x| < 5$ then $\frac{x+3}{x} > \frac{2}{5}$.

Assume that $|x| < 5$. We want to show that $\frac{x+3}{x} > \frac{2}{5}$

We can write $\frac{x+3}{x} > \frac{2}{5}$ as follows:

$$\frac{x+3}{x} - \frac{2}{5} > 0.$$

So, $\frac{5(x+3) - 2x}{5x} > 0.$

Then we ~~(simplify)~~, compute the numerator

$$\frac{5x + 15 - 2x}{5x} > 0$$

$$\frac{3x + 15}{5x} > 0$$

Then, we simplify:

$$3x + 15 > 0$$

$$\frac{15}{3} \overset{>}{\cancel{<}} \frac{-3x}{3} = 5 > -x$$

Since $|x| < 5$, then $5 > |-x|$ i.e. $5 > |x|$.

Hint: $\frac{a}{b} > \frac{c}{d}$ $(*)$ <u>method 1</u>

If $bd > 0$ then $(*)$ is equivalent to $ad > bc$.

If $bd < 0$ then $(*)$ is equivalent to $ad < bc$.

<u>Method 2</u>

$(*)$ is equivalent to

$$\frac{a}{b} - \frac{c}{d} > 0$$

equivalent to $\frac{ad - bc}{bd} > 0$

Then show that $ad - bc$ and bd have the same sign.

5/14/2025

Proof by Contrapositive.

$(P \Rightarrow Q) \iff (\sim Q \Rightarrow \sim P)$

ERROR

Ex: Show that if $\underbrace{n \in \mathbb{N}}_{P}$ then $\underbrace{n^2+4 \text{ is not a perfect square.}}_{Q}$

Assume n^2+4 is a perfect square.

Ex: Show that if $3 \nmid n$ then $n^2 \equiv 1 \pmod 3$.

Direct proof:

Assume $3 \nmid n$. We have two cases.

Case 1: $n \equiv 1 \pmod 3$

$n-1 = 3k$ for some $k \in \mathbb{Z}$

$n = 3k+1$

There is a remainder of 1.

Case 2: $n \equiv 2 \pmod 3$

$n-2 = 3l$ for some $l \in \mathbb{Z}$.

$n = 3l+2$.

There is a remainder of 2.

$n = 3k+1$

$(3k+1)^2 - 1 = 3$

$9k^2+6k+1-1 = 31$

$9k^2+6k = 3'$

$3k(3k+2) = 31$.

$3k+2 = 1$

$3k = -1$

$n = 3l+2$

$(3l+2)^2 - 1 = 3l$

$9l^2+12l+4-1 = 3l$

$9l^2+12l+3 = 3l$

$\frac{3(3l^2+4l+1)}{3} = \frac{3l}{3}$

$l = 3l^2+4l+1$

CamScanner

Assume $3 \nmid n$. We have two cases.

Case 1: $n \equiv 1 \pmod 3$

This can be written as $n-1 = 3k$ for some $k \in \mathbb{Z}$.

So, $n = 3k+1$. There is a remainder of 1.

continuation

Case 2: $n \equiv 2 \pmod 3$

This can be written as $n-2 = 3l$ for some $l \in \mathbb{Z}$.

So, $n = 3l+2$. There is a remainder of 2.

continuation

we want to show $n^2 - 1 = 3m$ for some $m \in \mathbb{Z}$.

We have $n = 3k+1$.

So, $(3k+1)^2 - 1 = 3m$.

This is equivalent to $9k^2 + 6k = 3m$.

To further simplify,

$m = 3k^2 + 2k$.

We have found $m = 3k^2 + 2k$.

We want to show $n^2 - 1 = 3r$ for some $r \in \mathbb{Z}$.

we have $n = 3l + 2$.

So, $(3l+2)^2 = 3r$.

This is equivalent to $9l^2 + 12l + 3 = 3r$.

To further simplify,

$r = 3l^2 + 4l + 1$.

We have found $r = 3l^2 + 4l + 1$

$a \equiv b \pmod{n} \iff a^2 \equiv b^2 \pmod{n}$

$2a \equiv 2b \pmod{n}$

$a \equiv b \pmod{n}$

$2 \equiv 2 \pmod{n}$

EX: What is the remainder of 3^{2025} in the division by 20?

$3^4 = 81 \equiv 1 \pmod{20}$

$(3^4)^{506} \equiv 1^{506} = 1 \pmod{20}$

$3^{2024} \equiv 1 \pmod{20}$

$3^{2025} \equiv 3 \pmod{20}$

5/16/2025

Absolute value

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties:

- $|xy| = |x||y|$ for any $x, y \in \mathbb{R}$.
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for any $x, y \in \mathbb{R}$, $y \neq 0$.
- Triangle inequality:
  $|x+y| \leq |x| + |y|$ for any $x, y \in \mathbb{R}$.
- $|x^n| = |x|^n$ for any $x \in \mathbb{R}$, $n \in \mathbb{N}$
- $|x^{-n}| = |x|^{-n}$ for any $x \in \mathbb{R}$, $x \neq 0$ and $n \in \mathbb{N}$.
- $|x| \geq x$ and $|x| \geq -x$ for any $x \in \mathbb{R}$.

Ex: Let $x, y \in \mathbb{R}$. Prove that

(a) $|-x| = |x|$.

(b) $|x^2| = x^2$

(c) $|x^5| = |x|^3$

(d) $|x-y| \geq |x| - |y|$.

Assume that $x, y \in \mathbb{R}$. Show that, $|-x| = x$, $|x^2| = x^2$, $|x^3| = |x^3|$, and $|x-y| \geq |x| - |y|$.

(a) Show $|-x| = |x|$.

Since $-x = (-1)x$, we can write it as follows.

$$\begin{aligned} |-x| &= |-1||x| \\ &= (1)|x| \\ &= |x|. \end{aligned}$$

Therefore, $|-x| = |x|$.

(b) Show $|x^2| = x^2$,

By the definition of absolute value,

$$|x^2| = \begin{cases} x^2 & \text{if } x^2 \geq 0, \\ -x^2 & x^2 < 0. \end{cases}$$

because $x^2 \geq 0$, we have $|x^2| = x^2$.

(c) show $|x^3| = |x|^3$.

Consider two cases: $x \geq 0$ and $x < 0$.

Case 1: $x \geq 0$

Then $x^3 \geq 0$. Thus $|x^3| = x^3$.

Because $x \geq 0$, we have $|x| = x$.

So $|x|^3 = x^3$. Therefore, $|x^3| = |x|^3$

Triangle Inequality

$a = x^2$ $b = -5$

$|a+b| \leq |x^2| + |-5|$,

triangle

$\underline{|x||x| + |5|}$

$|x-1| < 2$

$2(|x| + |-1|)$

$|x-1| \leq 2|x| + 2|-1|$

$|x-1|$

Extra Credit:

Show that $|x| \geq x$ and $|x| \geq -x$ for any $x \in \mathbb{R}$.

May 19, 2025

Examples:

① Let $x \in \mathbb{R}$ be such that $|x| \leq 2$.
Show that $|2x^2 - 7x - 5| \leq 30$.

Assume $x \in \mathbb{R}$ and $|x| \leq 2$.
we can rewrite $|2x^2 - 7x - 5|$ as follows:

$$|2x^2 + (-7x) + (-5)|.$$

Applying Triangle Inequality:

$|a+b| \leq |a| + |b|$, for $a = 2x^2 + (-7x)$ and $b = -5$,

we get:

$$|2x^2 + (-7x) + (-5)| \leq |2x^2 + (-7x)| + |(-5)|.$$

Applying Triangle Inequality once more to the right hand side for $a = 2x^2$ and $b = (-7x)$, we get

$$|2x^2 + (-7x)| + |(-5)| \leq |2x^2| + |(-7x)| + |(-5)|.$$

By property of Absolute Value,

$|xy| = |x||y|$ for any $x, y \in \mathbb{R}$, we can further simplify the right-hand side to

$$|2x^2| + |(-7x)| + |(-5)| \leq |2||x^2| + |-7||x| + |-5|.$$

Since $|x^2| = |x|^2$, the right-hand is as follows:

$$|2||x^2| + |-7||x| + |-5| \leq |2||x|^2 + |-7||x| + |-5|$$

we have $|x| \leq 2$, therefore the right-hand side will be:

$$|2|(2)^2 + |-7|(2) + |-5| = 2(4) + 7(2) + 5 = 27.$$

Hence, $27 < 30$.

② Let $x \in \mathbb{R}$ be such that $|x-1| \leq 2$.
Show that $|x^2-5| < 15$.

~~Try~~ The hypothesis is
$$p: (x \in \mathbb{R}) \wedge (|x-1| \leq 2).$$
The conclusion is
$$Q \quad (|x^2-5| < 15)$$
we need Q to show $p \Rightarrow Q$.

Assume $x \in \mathbb{R}$ and $|x-1| \leq 2$. Show that
$|x^2-5| < 15$.

We have $|x-1| \leq 2$. Add 1 to both sides:
$$|x-1|+1 \leq 2+1.$$
We now have:
$$|x-1|+1 \leq 3 \qquad (1)$$
Applying Triangle inequality $|a+b| = |a|+|b|$ (*) for
$a = x-1$ and $b=1$, we get:
$$|(x-1)+1| \leq |x-1|+1 \qquad (2)$$
By (1) and (2), simplifying ~~LHS (2)~~, we have:
$$|x| \leq 3.$$

We want to show $|x^2-5| < 15$.
Applying Triangle inequality (*) for $a = x^2$ and $b = -5$,
we get to
$$|x^2-5| \leq |x^2| + |-5|$$
by Product Rule, $|x^2| = |x||x|$ for any $x \in \mathbb{R}$.

$n^2+1 = 5k$

May 20, 2025

Proof by contrapositive:

$(\underbrace{P \Rightarrow Q}_{\text{direct}})$ is equivalent to $(\underbrace{\sim Q \Rightarrow \sim P}_{\text{contrapositive}})$

Proof by contradiction:

$(P \Rightarrow Q)$ is equiv. to $(P \wedge \sim Q) \Rightarrow$ contradiction

Q is true is equivalent to $(\sim Q \Rightarrow$ contradiction$)$.

Ex: Prove that $\sqrt{6}$ is irrational.

Suppose / Assume by contradiction that $\sqrt{6}$ is rational.

So, $\sqrt{6} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b \neq 0$. We can assume that the fraction $\frac{a}{b}$ is in simplified form. (We can assume that $GCD(a,b)=1$).

By squaring both sides, we have/get/obtain:

$$6 = \frac{a^2}{b^2}.$$

Multiply both sides by b^2, we get

$$6b^2 = a^2. \quad (1)$$

Because $a^2 = 2(3b^2)$ is an even number, a must be an even number. So, $a = 2c$ for some $c \in \mathbb{Z}$.

Substituting $a = 2c$ into (1),

we get $6b^2 = (2c)^2 = 4c^2$,

which is equivalent to

$$3b^2 = 2c^2.$$

So, ~~(Because)~~ $3b^2$ is an even number.

So, b^2 is an even number.

So, b is an even number.

So, $b = 2d$, for some $d \in \mathbb{Z}$.

Another example: show that $\sqrt{2} + \sqrt{3}$ is irrational.

Suppose by contradiction that $r = \sqrt{2} + \sqrt{3}$ is rational. Then $r^2 = 5 + 2\sqrt{6}$ is also rational. Then $\sqrt{6} = \frac{r^2 - 5}{2}$ is also rational. This contradicts the fact that $\sqrt{6}$ is irrational.

May 23, 2025

Quantified Statements

Statements that start with "There exists" or "For all".

Ex: There exists $n \in \mathbb{N}$ such that $n^2 \equiv 6 \pmod{10}$. (True)

Ex: For all $n \in \mathbb{N}$, $2^n - 1$ is prime. (False)

Notations:

$\exists n \in \mathbb{N}$ s.t. $n^2 \equiv 6 \pmod{10}$.

error

$\forall$ ~~*~~ $n \in \mathbb{N}$, $2^n - 1$ is prime.

$\exists! n \in \mathbb{N}$ s.t. $n^2 \equiv 6 \pmod{10}$

There exist only one.

$\exists\, n, k \in \mathbb{N}$ s.t. $n^2 \equiv 6 \pmod{10}$ and $k^2 \equiv 6 \pmod{10}$.

$$\underbrace{\exists\, n_1, n_2, n_3, n_4, n_5 \in \mathbb{N} \text{ s.t. } \forall k \in \{1, \dots, 5\}, n_k^2 \equiv 6 \pmod{10}}_{\text{nested quantified statement}}$$

EXISTENTIAL STATEMENT

$\exists n \in \mathbb{N}$ s.t. $n^2 \equiv 6 \pmod{10}$

existence symbol

s.t.: such that / so that punctuation

UNIVERSAL STATEMENT

$\boxed{\forall} n \in \mathbb{N} \boxed{,}$ $2^n - 1$ is prime

universality symbol

, : punctuation

6.6 Exercise

7. (a) It is true ~~because then~~

(b) True → False

(c) True

(d) False

$\exists n \in A$ s.t. $\underbrace{P(n)}_{\text{statement}}$: There exist $n \in A$ such that / so $P(n)$.

How to prove?

We need to find $n \in A$ such that $P(n)$ is true.

$\forall n \in A$, $P(n)$: For all / each / every $n \in A$, $P(n)$ is true.

How to prove?

Pick any $n \in A$. We will show that $P(n)$ is true.

Equivalent:

For each $n \in A$, we will show that $P(n)$ is true.

7(a): We need to find $x \in \mathbb{Z}$ such that $\underbrace{\text{there exists } y \in \mathbb{Z} \text{ such that } x + y = 3}$.

Pick $x = 0$. We need to find $y \in \mathbb{Z}$ such that $0 + y = 3$.

We can choose $y = 3$. ■

7(c.) For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $x + y = 3$. ~~is true~~!

Pick any $x \in \mathbb{Z}$. We will show that there exists $y \in \mathbb{Z}$ such that $x + y = 3$. ~~is true~~.

(~~We can choose $y = 3 - x$~~) We need to find $y \in \mathbb{Z}$ such that $x + y = 3$.

We can choose $y = 3 - x$.

$\exists x \in A$ s.t. $P(x)$.

Negation:

$\forall x \in A$, $\sim P(x)$.

$\forall x \in A$, $P(x)$.

Negation:

$\exists x \in A$ s.t. $\sim P(x)$.

$\forall x \in A$, ($\exists y \in B$ s.t. $P(x,y)$).

Negation:

$\exists x \in A$ s.t. $\forall y \in B$, $\sim P(x,y)$.

May 28, 2025

6.4 $\forall a,b\, \mathbb{Z}$ if $3|(a^2+b^2)$ then $3|a$ and $3|b$.

$\downarrow$

Pick $a,b \in \mathbb{Z}$. We want to show that if $3|(a^2+b^2)$ then $3|a$ and $3|b$.

$\hookrightarrow$ can be replaced by $3|(a^2+b^2) \Rightarrow ((3|a) \wedge (3|b))$.

Assume $3|(a^2+b^2)$. We need to show $3|a$ and $3|b$.

To do this, we will first prove ~~that~~ the following lemma.

$\forall n \in \mathbb{Z}$, $n^2 \equiv 0$ or $1 \pmod 3$.] lemma

Pick $n \in \mathbb{Z}$. We need to show that $n^2 \equiv 0$ or $n^2 \equiv 1 \pmod 3$.

We consider 3 cases $n \equiv k \pmod 3$, for $k=0,1,2$.

Case 1: $n \equiv 0 \pmod 3$

Then $n^2 \equiv 0^2 = 0 \pmod 3$.

Case 2: $n \equiv 1 \pmod 3$

Then $n^2 \equiv 1^2 = 1 \pmod 3$.

Case 3: $n \equiv 2 \pmod 3$

Then $n^2 \equiv 4 = 1 \pmod 3$.

Therefore, the lemma has been proven.

Let us go back to the problem.

By the lemma,

$a^2 \equiv 0$ or $1 \pmod 3$,

$b^2 \equiv 0$ or $1 \pmod 3$.

So, $a^2+b^2 \equiv 0$ only in the case $a^2 \equiv 0$ and $b^2 \equiv 0 \pmod 3$.

So, ~~$a \equiv 0$~~ $3|a^2$ and $3|b^2$. So, $3|a$ and $3|b$.

9.) Consider the following logical statement: · True

$\forall n \in \mathbb{N}, \exists x \in \mathbb{R}$ s.t. $x^2 > n$. - For all n is a Natural number, there exists a real number x such that x^2 is greater than n.

Pick any $n \in \mathbb{N}$. We will show that there exists $x \in \mathbb{R}$ s.t. $x^2 > n$.

We need to find $x \in \mathbb{R}$ such that $x^2 > n$.

~~Since $x^2 > n$~~, Let us ~~We~~ can choose $x = n+1$. We have:

$\cancel{(n+1)^2 > n}$ $x^2 = (n+1)^2 = n^2 + 2n + 1 = n^2 + n + n + 1$.

$\cancel{n^2 + 2n + 1 > n}$. $> 0 + 0 + n + 0 = n$

Let us choose $x = n+1$. We have:

$x^2 = (n+1)^2 = n^2 + 2n + 1 = n^2 + n + n + 1 > 0 + 0 + n + 0 = n$.

10. Prove or disprove the statement:

$$\forall x < -\frac{1}{4}, \frac{x}{x-1} > \frac{1}{2}$$

6/2/2025

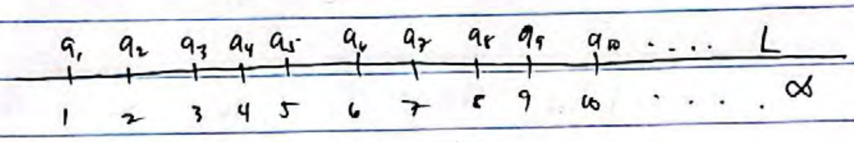
June 1, 2025

* Don't use contradiction right away

Limit of a sequence

* Definition: A number $L \in \mathbb{R}$ is the limit of a sequence (a_n) if

$$\forall \varepsilon > 0, \underbrace{\exists N \in \mathbb{N} \text{ s.t. } \underbrace{\forall n \in \mathbb{N}, n > N \Rightarrow |a_n - L| < \varepsilon}_{Q(N,\varepsilon)}}_{P(\varepsilon)}.$$

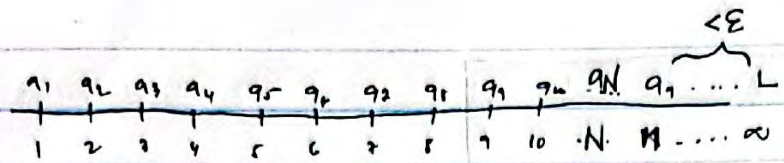


- $a_n \to L$ as $n \to \infty$
- $\lim_{n\to\infty} a_n = L$
- $\lim a_n = L$

are equivalent notations.

In full sentence, we write:

"The sequence (a_n) $\underbrace{\text{tends to}}_{\text{approaches / goes to}}$ L as n $\underbrace{\text{tends to}}_{\text{approaches / goes to}}$ infinity."



Ex: Let $a_n = \frac{2n+1}{3n+2}$.

Prove that $\lim a_n = \frac{2}{3}$

Pick any $\underbrace{\varepsilon \in (0,\infty)}_{\varepsilon > 0}$. We want to find $N \in \mathbb{N}$ s.t.

$$\forall n \in \mathbb{N},\ n > N \Rightarrow \left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| < \varepsilon.$$

Assume $n \in \mathbb{N}$ and $n > N$. We have

$$\frac{2n+1}{3n+2} - \frac{2}{3} = \frac{3(2n+1) - 2(3n+2)}{3(3n+2)} = \frac{-1}{3(3n+2)}.$$

Taking the absolute value of both sides, we get

$$\left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| = \frac{|-1|}{|3(3n+2)|} = \frac{1}{3(3n+2)}.$$

Because $n > N$,

$$\left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| = \frac{1}{3(3n+2)} < \frac{1}{3(3N+2)}.$$

To ensure that $\left| \frac{2n+1}{3n+2} - \frac{2}{3} \right| < \varepsilon$, we need to pick $N \in \mathbb{N}$ such that $\frac{1}{3(3N+2)} < \varepsilon$.

This inequality is equivalent to

$$3(3N+2) > \frac{1}{\varepsilon},$$

which is equivalent to

$$3N+2 > \frac{1}{3\varepsilon},$$

which is equivalent to

$$N > \frac{1}{3}\left(\frac{1}{3\varepsilon} - 2\right).$$

Floor function :

$\lfloor a \rfloor$ is the largest integer that is smaller than or equal to a.

$\lfloor 2.7 \rfloor = 2$ $\lfloor 5 \rfloor = 5$

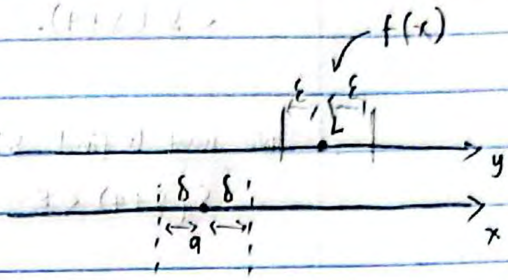
$\lfloor 10.2 \rfloor = 10$ $\lfloor -5 \rfloor = -6$

We pick $N = \left\lfloor \frac{1}{3}\left(\frac{1}{3\varepsilon} - 2\right) \right\rfloor + 1$

* We pick $N = \max\left\{ \left\lfloor \frac{1}{3}\left(\frac{1}{3\varepsilon} - 2\right) \right\rfloor + 1 , 1 \right\}$. ← More complete.

$$\begin{aligned}
|x^2-4| &= |(x-2)(x+2)| \\
&= |(x-2)|\,|(x+2)| \\
&\leq \delta\,|x+2| \\
&= \delta\,|(x-2)+4| \\
&\leq \delta\,(|x-2|+|4|) \\
&< \delta\,(\delta+4)
\end{aligned}$$

June 4, 2025



Definition:

Let f be a function of real variable.

We say that f converges to L as x tends to a if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R}, a-\delta \le x \le a+\delta \Rightarrow L-\varepsilon \le f(x) \le L+\varepsilon,$$

* For any positive ε, we can find a positive number δ, such that as long as x is within δ away from a, f(x) will be within ε away from L.

We (need) to write like like this:
can

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \underbrace{(a-\delta < x < a+\delta) \wedge (x \neq a)}_{0 < |x-a| < \delta} \Rightarrow \underbrace{L-\varepsilon < f(x) < L+\varepsilon}_{|f(x)-L| < \varepsilon}.$$

Ex: Prove that the $\lim_{x \to 2} x^2 = 4$.

Answer:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x-2| < \delta \Rightarrow |x^2-4| < \varepsilon.$$

Pick any $\varepsilon > 0$. We want to find $\delta > 0$ such that $\boxed{0 < |x-2| < \delta \Rightarrow |x^2-4| < \varepsilon.}$ new line

$$0 < |x-2| < \delta \Rightarrow |x^2-4| < \varepsilon.$$

Assume $0 < |x-2| < \delta$. [~~we want to show $|x^2-4| < \varepsilon$~~] ERROR

We have,

$$|x^2-4| = |(x-2)(x+2)|$$
$$= |(x-2)||(x+2)|$$
$$< \delta |x+2|$$
$$= \delta |(x-2)+4|$$

$$\le \delta\,(|x-2| + |4|)$$
$$< \delta\,(\delta+4).$$

We want to find $\delta > 0$ such that

$$\delta(\delta+4) < \varepsilon.$$

Choose $\delta = \frac{1}{2}\min\left\{1, \frac{\varepsilon}{5}\right\}$.

Then $\delta \le \frac{1}{2} < 1$ and $\delta \le \frac{1}{2}\frac{\varepsilon}{5} < \frac{\varepsilon}{5}$.

We have

$$\delta(\delta+4) < \delta(1+4) = 5\delta < 5\frac{\varepsilon}{5} = \varepsilon.$$

June 6, 2025

Example of proving the limit of a sequence doesn't exist.

$a_n = (-1)^n$

Show that $\lim a_n$ doesn't exist.

<u>Method 1:</u>

Suppose by contradiction that the limit exists.

Let $L = \lim a_n$.

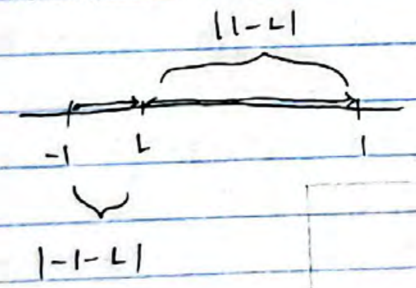
We have

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ st. $n \in \mathbb{N}$, $n > N \Rightarrow |a_n - L| < \varepsilon$.

We have

$$|a_n - L| = |(-1)^n - L| = \begin{cases} |1-L| & \text{if } n \text{ is even} \\ |-1-L| & \text{if } n \text{ is odd} \end{cases}$$

Scratchwork



Pick $\varepsilon = \frac{1}{3}$. There exists $N \in \mathbb{N}$ such that

$\forall n \in \mathbb{N}$ (and), $n > N \Rightarrow |(-1)^n - L| < \frac{1}{3}$.

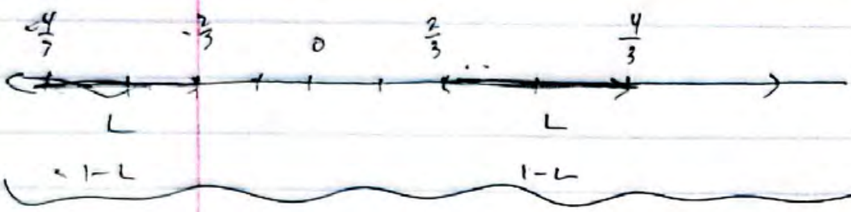
For $n = 2N$, we have $|(-1)^{2N} - L| < \frac{1}{3}$.

In other words, $|1 - L| < \frac{1}{3}$. ①

For $n = 2N+1$ ..... $|-1 - L| < \frac{1}{3}$. ②

(1) implies $-\frac{1}{3} < 1-L < \frac{1}{3}$, which implies $\frac{2}{3} < L < \frac{4}{3}$.

(2) implies $-\frac{1}{3} < -1-L < \frac{1}{3}$, which implies $-\frac{4}{3} < L < -\frac{2}{3}$



We have

$$L < -\frac{2}{3} < \frac{2}{3} < L.$$

This is a contradiction. ■

Mathematical Induction

$P(n)$ is a statement depending on n.

> If
> - $P(a)$ is true,
> - $\forall k \in \mathbb{Z}$ and $k \geq a$, $P(k) \Rightarrow P(k+1)$ is true,
>
> then $P(k)$ is true for all $k \in \mathbb{Z}$, $k \geq a$.

Ex: Show that for all $n \geq 5$, $2^n > n^2$.

$P(n)$: "$2^n > n^2$"

$P(5)$: "$32 > 25$".

$P(5)$ is true ← the base case.

We want to show $\forall k \geq 5$, $P(k) \Rightarrow P(k+1)$.

Pick $k \in \mathbb{Z}$, $k \geq 5$.

Want to show $P(k) \Rightarrow P(k+1)$.

Assume $P(k)$ is true, we want to show $P(k+1)$ also true.

have: $2^k > k^2$

show: $2^{k+1} > (k+1)^2$

$$2^{k+1} = 2 \cdot 2^k > 2k^2$$

$$\vee$$

$$(k+1)^2 = k^2 + 2k + 1$$

June 9, 2025

Ex: Show that $2^n > n^2$ $\forall n \in \mathbb{N}, n \geq 5$.

can be interchanged

We show by induction on $n \geq 5$ that

$$2^n > n^2. \qquad (1)$$

For $n = 5$, we have

$$2^5 = 32 > 25 = 5^2.$$

So, (1) is true for $n = 5$.

Suppose that (1) is true for $n = k$ for som $k \geq 5$.

We have

$$2^k > k^2. \qquad (2)$$

We want to show that

$$2^{k+1} > (k+1)^2. \qquad (3).$$

We have / Because of (2), we have

$$2^{k+1} = 2 \cdot 2^k > 2k^2$$

To prove (3), it is sufficient to show that

$$2k^2 \geq (k+1)^2. \qquad (4)$$

(4) is equivalent to

$$k^2 \geq 2k+1 \qquad (5)$$

We have

$$k^2 - 2k = k(k-2) \geq 5 \cdot 3 = 15 \geq 1.$$

Therefore, (5) is true.

26. Let $a_0, a_1, a_2 \ldots$ be a sequence recursively defined as $a_0 = 2$, $a_1 = 1$, and

$$a_n = a_{n-1} + 6a_{n-2} \quad \text{for } n \geq 2.$$

Prove by induction that

$$a_n = (-2)^n + 3^n \quad \text{for all } n \geq 0.$$

We show by induction that on $n \geq 0$ that

$$a_n = (-2)^n + 3^n.$$

Ex: Let $a_1, a_2, a_3, \ldots$ be a sequence defined recursively as $a_1 = 3$ and for all $n \geq 2$,

$$a_n = 2a_{n-1} - 1.$$

Prove that $a_n = 2^n + 1$ for all $n \in \mathbb{N}$.

We show by induction ~~(that)~~ on $n \geq 1$ that

$$a_n = 2^n + 1. \quad (1)$$

For $n = 1$, we have

$$\cancel{a_2 = 2^2+1 = 5.}$$

$$\cancel{a_1 = 2^1+1 = 3.}$$

$$a_1 = 3 = 2^1 + 1.$$

So, (1) is true for $n=1$.

Suppose that (1) is true for $n=k$ for some $k \geq 1$.

We have

$$a_k = 2^k + 1. \quad (2)$$

We want to show that

$$a_{k+1} = 2^{k+1} + 1. \quad (3)$$

formula

Because of the recursive ~~(sequence)~~, we have

$$a_{k+1} = 2a_k - 1.$$

Because of (2), we have

$$a_{k+1} = 2(2^k+1) - 1 = 2^{k+1} + 1.$$

Therefore, (3) is true.

June 11, 2025

Chapter 7

23. Let $a \in \mathbb{R}$ s.t. $a + \frac{1}{a} \in \mathbb{Z}$. Prove that $a^n + \frac{1}{a^n} \in \mathbb{Z}$ for any $n \in \mathbb{N} \cup \{0\}$.

$\forall$

~~Pf~~

Scratch : $P(0)$: $a^0 + \frac{1}{a^0} \in \mathbb{Z}$ ✓

$(a+b)^2 = a^2+b^2+2ab$

$P(1)$: $a^1 + \frac{1}{a^1} \in \mathbb{Z}$ ✓

$P(2)$: $a^2 + \frac{1}{a^2} \in \mathbb{Z}$ not so sure

$a^2 + \frac{1}{a^2} = (a+\frac{1}{a})^2 - 2 \in \mathbb{Z}$ ✓

$P(3)$: $a^3 + \frac{1}{a^3} \in \mathbb{Z}$ ✓

$a^{k-1} + \frac{1}{a^{k-1}}$

$a^3 + \frac{1}{a} + a + \frac{1}{a^3}$

$\frac{1}{a} + \frac{a}{1} = \frac{1+a+a}{a} = \frac{1+2a}{a}$

$a^3 + \frac{1}{a^3} + \frac{1}{a} + a$

$a^k + \frac{1}{a^k}$

$\cancel{\frac{a^k a^0}{a^0}} + \frac{a^0}{a^k a^0}$

$\left(a^3 + \frac{1}{a^3}\right) + \left(a + \frac{1}{a}\right)$

$\frac{a^k \cdot a}{a} + \frac{a}{a^k}$

$a^4 + \frac{1}{a^2} + a^2 + \frac{1}{a^4}$

$\frac{a^k}{a} + \frac{a}{a^k}$

$\left(a^4 + \frac{1}{a^4}\right)$ then $\left(a^2 + \frac{1}{a^2}\right) - \left(a^2 + \frac{1}{a^2}\right)$

CamScanner

We show by induction on $n \geq 0$ that

$$a^n + \frac{1}{a^n} \in \mathbb{Z}. \quad (1)$$

For $n=0$, we have

$$a^0 + \frac{1}{a^0} = 2 \in \mathbb{Z}.$$

So, (1) is true for $n=0$.

For $n=1$, we have

$$a^1 + \frac{1}{a^1} = a + \frac{1}{a}$$

which is an integer because of the hypothesis.

For $n=2$, we have

$$a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$$

which is an integer because $a + \frac{1}{a}$ is an integer.

Suppose (1) is true for all $n \leq k$ for some $k \geq 2$.

We show that (1) is true for $n = k+1$.

$$a^{k+1} + \frac{1}{a^{k+1}} = \left(a^k + \frac{1}{a^k}\right)\left(a + \frac{1}{a}\right) - \left(\frac{a^k}{a} + \frac{a}{a^k}\right)$$

$$= \left(a^k + \frac{1}{a^k}\right)\left(a + \frac{1}{a}\right) - \left(a^{k-1} + \frac{1}{a^{k-1}}\right).$$

This is an integer because $a^k + \frac{1}{a^k}$, $a + \frac{1}{a}$, $a^{k-1} + \frac{1}{a^{k-1}}$ are all integers.

Chapter 9 : Relation

A relation on a set A is a subset of the set $A \times A$.

Here, $A \times A$ $\{(x,y) : x \in A,\ y \in A\}$.

is called a cartesian Product.

Ex:

$A = \{2,3,4\}$

$B = \{1,2\}$

$A \times B = \{(2,1),(2,2),(3,1),(3,2),(4,1),(4,2)\}$

This is the Cartesian Product of A and B.

$A \times A = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$

$R = \{(2,3),(2,4),(3,4)\} \subseteq A \times A.$

So, R is a relation.

$2R3$ means $(2,3) \in R$ $\quad$ $3 \not R 3$ because $(3,3) \notin R$.

$2R4$ means $(2,4) \in R$ $\quad$ $xRy \Rightarrow$ ~~iff~~, $x < y$, for this specific example.

$\cdot\ xRy \Leftrightarrow (x,y) \in R$.

6/13/2025

Definition: Let R be a relation on A.

- R is reflexive if xRx for all $x \in A$. $\rightarrow \forall x \in A,\ xRx$.
- R is symmetric if $xRy \Rightarrow yRx$
- R is transitive if $(xRy) \wedge (yRz) \Rightarrow xRz$.
- R is total if $\forall x,y \in A$, $(xRy) \vee (yRx)$
- R is antisymmetric if $(xRy) \wedge (yRx) \Rightarrow x = y$.
- R is dense if $\forall x,y \in A$, $\exists z \in A$ such that $(xRz) \wedge (zRy)$.

Ex: On the $\mathbb{N}$, consider the following relation:

$mRn \Leftrightarrow \gcd(m,n) > 1.$

$\mathbb{N} = \{1,2,3,4,5,\ldots\}$

$R = \{(6,8),\ (8,10),\ (9,21),\ (10,25),\ (4,10),\ (4,14),\ (12,18),\ (6,9),\ (14,21),\ (8,$ [illegible]

- R is not reflexive because $1 \in \mathbb{N}$ and $1 \not R 1$.
- R is symmetric

  Pick any $m, n \in A$. We want to show that

  $$mRn \Rightarrow nRm.$$

  Assume mRn. We need to show nRm.

  Because ~~we have~~ mRn, we have

  $$\gcd(m,n) > 1.$$

  We know that

  $$\gcd(m,n) = \gcd(n,m).$$

  So,

  $$\gcd(n,m) = \gcd(m,n) > 1.$$

  Therefore, $mRn \Rightarrow nRm$.
- R is not transitive because $12 R 14$ and $14 R 7$ but $12 \not R 7$.
- R is not total because $7 \not R 11$ and $11 \not R 7$.

  ~~Pick any $m, n \in A$. We want to show that~~

  $(mRn) \vee (mRk)$.
- R is not antisymmetric because $(4R6) \wedge (6R4)$ but $4 \neq 6$.
- R is not dense because $1 \not R 1$.

June 16, 2025 (Math 301)

Equivalence Relations

An equivalence relation is a non-empty relation that is reflexive, symmetric, and transitive.

Ex: $A = \mathbb{R}$ $R = \{(a,b) : a,b \in \mathbb{R}, a=b\}$

$aRb \Leftrightarrow a=b$ $= \{(a,a) : a \in \mathbb{R}\}$

R is an equivalence relation.

Ex: $A = \mathbb{R}$ $R = \{(a,b) : a,b \in \mathbb{R}, a \leq b\}$

$aRb \Leftrightarrow a \leq b$

reflexive (v), transitive (v), symmetric (x)

anti symmetric (v)

R is not an equivalence relation.

Ex: $A = \mathbb{N}$

aRb if $2 \mid (a+b)$

Is R an equivalence relation?

→ We need to check whether R is reflexive, transitive, and symmetric.

- Check if R is reflexive.
- Check if R is symmetric.
- Check if R is transitive.

Check if R is reflexive

Pick any $a, b \in \mathbb{N}$. We want to show aRb iff and only if $2 \mid (a+b)$.

Assume

$1 R 3 R 5 R 7 R 9 R \dots$

$1, 3, 5, 7, 9, \dots$ all "R" each other.

$2, 4, 6, 8, 10, \dots$ all "R" each other.

Let R be an equivalence relation on A.

For any $x \in A$, the equivalence class of A that contains x is

$[x] = \{ y \in A : xRy \}$.

In the previous example,

$[1] = \{1, 3, 5, 7, 9, \dots\}$

$[3] = \{1, 3, 5, 7, 9, \dots\}$

$[5] = \{1, 3, 5, 7, 9, \dots\}$

$[2] = \{2, 4, 6, 8, 10, 12, \dots\}$

$[1] \cap [2] = \phi$

$\boxed{3}$ is a representative of the equivalence class $\{1, 3, 5, 7, \dots\}$.

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Ex: $A = \mathbb{N}$

$xRy \iff x \equiv y \pmod 6$ $\quad$ $\cdot (x-x=0,\ 6|0)$

· reflexive because $(x-x=0|6)$ for any $x \in \mathbb{N}$.

· symmetric because $(x-y)|6$ imply $(y-x)|6$.

$6|(x-y)$ implies $6|(y-x)$.

$[6] = \{6, 12, 18, 24, \ldots\}$

$[2] = \{2, 8, 14, 20, \ldots\} = [8] = [14] = \ldots$

June 18, 2025

Math 301

$aRb \iff 7a^2 \equiv 2b^2 \pmod 5$

$aRb \iff 2a^2 + 5a^2 \equiv 2b^2 \pmod 5$

$\iff 2a^2 \equiv 2b^2$

$\iff 2a^2 - 2b^2 = 5k$

$\iff 2(a^2-b^2) = 5k$

$\iff 2(a^2-b^2) = 5 \cdot 2l$

$\iff a^2 - b^2 = 5l$

$\iff a^2 \equiv b^2 \pmod 5$

$aRb \iff a^2 \equiv b^2 \pmod 5$

equivalence class

$[0] = \{0, \pm 5, \pm 10, \pm 15, \ldots\}$ $\quad a \equiv 0 \pmod 5$

$[1] = \{\pm 1, \pm 4, \pm 6, \pm 9, \pm 11, \ldots\}$ $\quad a \equiv \pm 1 \pmod 5$

$[2] = \{\pm 2, \pm 3, \pm 7, \pm 8, \pm 12, \pm 13\}$ $\quad a \equiv \pm 2 \pmod 5$

$A = \{a \in \mathbb{Z} : a \equiv 0 \pmod 5\}$

$B = \{a \in \mathbb{Z} : a \equiv \pm 1 \pmod 5\}$

$C = \{a \in \mathbb{Z} : a \equiv \pm 2 \pmod 5\}$

To show A, B, C are all equivalence classes:

1) A, B, C are $\boxed{\text{distinct}}$/ disjoint

nothing in common

2) $A \cup B \cup C = \mathbb{Z}$

$A_R = \{(x,y) : x^2 + y^2 = R^2\}$ where $R \geq 0$

Functions

$f(x) = x^2$ is a function. $A = \mathbb{R}$

$R = \{(x,y) : x, y \in \mathbb{R},\ y = f(x)\} \subseteq \mathbb{R} \times \mathbb{R}$.

$x R y \iff y = f(x)$

$1 \not R 2$

$1 R 1$

$1 \not R x$ for all $x \neq 1$.

A function $f : A \to A$ is a relation on A.

$x R y \iff y = f(x)$

$R = \{(x,y) \in A \times A : y = f(x)\}$ (graph of f).

Not all relations are functions.

$f : \{1,2,3\} \to \{1,2,3\}$

$R = \{(1,1), (1,2), (2,3), (3,1)\}$ ← not a function

$f(1) = 1$

$f(1) = 2$

Rigorous Definition

Functions

A function $f : A \to B$ is a subset of $A \times B$ such that

$\forall x \in A,\ \exists!\ y \in B$ s.t. $(x,y) \in f$.

$\exists!$: exists uniquely

A: domain of f

B: co-domain of f

Ex: Check if the following are functions from ($\mathbb{Z}$ to $\mathbb{Z}$) $A = \mathbb{Z}$ to $B = \mathbb{Z}$:

1) $f = \{(n, n+1) : n \in \mathbb{Z}\}$

2) $f = \{(n^2, n) : n \in \mathbb{Z}\}$

3) $f = \{(n+1, n) : n \in \mathbb{Z}\}$

4) $f = \{(n, \frac{1}{1+n^2}) : n \in \mathbb{Z}\}$

1) $f = \{(n, n+1) : n \in \mathbb{Z}\}$

$\forall x \in A, \exists! \, y \in B$ s.t. $(x,y) \in f$.

Negation:

$\exists x \in A$ s.t. either there is no y such that $(x,y) \in f$ or there are more than one such y.

Notation:

$\exists x \in A$ st. $(\forall y \in B, (x,y) \notin f) \vee (\exists y, z \in B$ s.t. $y \neq z \wedge (x,y) \in f \wedge (x,z) \in f)$.

Pick any $x \in \mathbb{Z}$. We want to find $y \in \mathbb{Z}$ such that $(x,y) \in f$ and show that such y is unique.

Choose/Pick $y = x+1$. We have $(x,y) = (x, x+1) \in f$.

Suppose $z \in \mathbb{Z}$ and $(x,z) \in f$.

Then $z = x+1 = y$.

So, y is unique.

2.) $f = \{(n^2, n) : n \in \mathbb{Z}\}$

Choose $x = -1$. We show that for any $y \in \overset{\mathbb{Z}}{B}$, $(x,y) \notin f$.

Pick any $y \in \mathbb{Z}$. We want to show $(x,y) \in f$.

Suppose by contradiction that $(x,y) \in f$.

Then $(x,y) = (n^2, y)$ for some $n \in \mathbb{Z}$.

$x = n^2$

$\boxed{-1 = n^2}$ contradiction.

③ $f = \{(n+1, n) : n \in \mathbb{Z}\}$.

Pick any $x \in \mathbb{Z}$. We want to find and show that y is unique.

Choose $y = x-1$.

We have $(x,y) = (x, x-1) = (x+1, x) \in f$.

Suppose $z \in \mathbb{Z}$ and $(x,z) \in f$.

Then $z = x-1 = y$.

So, y is unique.

④ $f = \{(n, \frac{1}{n^2+1}) : n \in \mathbb{Z}\}$

Pick $x = 1$.

We want to show that for any $y \in \mathbb{Z}$, $(x,y) \notin f$.

Pick any $y \in \mathbb{Z}$. We want to show $(x,y) \notin f$.

Suppose, by contradiction, that $(x,y) \in f$.

Then $(x,y) = (n, \frac{1}{n^2+1})$ for some $n \in \mathbb{Z}$.

So, for $x = n = 1$, we have

$$y = \frac{1}{n^2+1} = \frac{1}{(1)^2+1} = \frac{1}{2}.$$

This is a contradiction because $\frac{1}{2} \notin \mathbb{Z}$.

Therefore, there is no such y ...

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Function $f : A \to B$ is a subset of $A \times B$ such that:

$\forall x \in A, \exists! y \in B$ s.t. $(x,y) \in f$. (*)

(*) is equivalent to:

1) $\forall x \in A, \exists y \in B$ s.t. $(x,y) \in f$.

2) $\forall x \in A, y, z \in B, (x,y) \in f \wedge (x,z) \in f \Rightarrow y = z$.

$(*) \iff (1) \wedge (2)$

Notation: Let f is a function from A to B.

If $(x,y) \in f$ then we write $y = f(x)$.

We say that y is the image of x under f.

Ex: $f = \{(1,2),(2,3),(3,0)\}$

$f(1) = 2$: 2 is the image of 1 under f.

$f(0)$ is undefined.

Let $f : A \to B$ be a function.

The set of all images under f is called the range of f.

$$\text{range}(f) = \{y \in B : \exists x \in A \text{ s.t. } y = f(x)\}.$$
$$= \{f(x) : x \in A\}$$

Another notation:

$$f(A) = \{f(x) : x \in A\}$$

Ex: $f = \{(1,2),(2,3),(3,0)\}$

range $(f) = \{2,3,0\}$.

domain $(f) = \{1,2,3\}$.

June 23, 2025

Math 301

Definition:

A function $f: A \to B$ is

- injective if

  $\forall x, y \in A$, $f(x) = f(y) \Rightarrow x = y$.

- surjective if

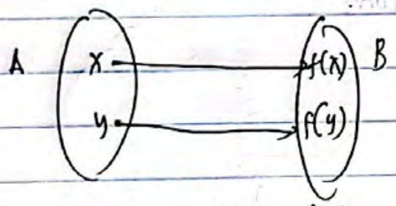
  $\forall y \in B$, $\exists x \in A$ s.t. $f(x) = y$.

- bijective if

  it is $\boxed{\text{both}}$ injective and surjective.

Ex: $f(x) = x^2$

$f(1) = f(-1)$, but $1 \neq -1$.



$x \neq y \Rightarrow f(x) \neq f(y)$

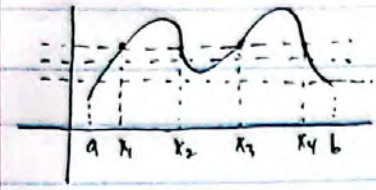
Injective: Two different elements in the domain have two different images.

$f_1: \mathbb{R} \to \mathbb{R}$, $f_1(x) = x^2$

$f_2: [0, \infty) \to \mathbb{R}$, $f_2(x) = x^2$

f_1 is not injective, but f_2 is injective.

Horizontal Line Test:



$f(x_1) = f(x_2) = f(x_3) = f(x_4)$

f is not injective

The function $f : A \to B$ (where A, B are subsets of $\mathbb{R}$) is injective if and only if any horizontal line intersects the graph of f at at most one point.

Ex: Show that $f : (1, \infty) \to \mathbb{R}$,

$$f(x) = \frac{2x+1}{x-1}$$

is injective.

Pick any $x, y \in (1, \infty)$. We want to show that

$$f(x) = f(y) \Rightarrow x = y.$$

That is:

$$\frac{2x+1}{x-1} = \frac{2y+1}{y-1} \Longrightarrow x = y. \qquad (1)$$

Assume that the LHS of (1) is true. We have,

$$\frac{2x+1}{x-1} = \frac{2y+1}{y-1}.$$

By multiplying each side by the product of their denominators, we get

$(2x+1)(y-1) = (2y+1)(x-1)$ which is equivalent to

$$2xy + y - 2x - 1 = 2yx + x - 2y - 1. \qquad (2)$$

By simplifying (2), we get

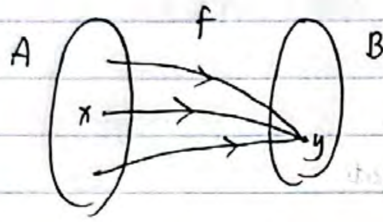
$$\left[\begin{array}{l} \cancel{2y + y = 2x + x,} \\ \cancel{3y = 3x,} \\ \cancel{y = x.} \end{array}\right] \qquad 3x = 3y, \quad x = y.$$

By simplifying (2), we get

$$3x = 3y \quad \text{which implies}$$

$$x = y. \blacksquare$$

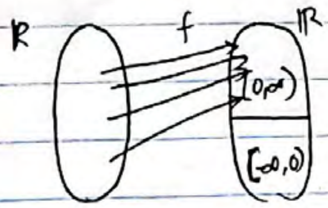
<u>Surjective</u>



$f: A \to B$ is surjective if every $y \in B$ is the image of some $x \in A$.

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

f is not surjective because there is no $x \in \mathbb{R}$ such that $f(x) = -1$.



Ex: $f: \mathbb{R} \to [0,\infty)$, $f(x) = x^2$.

Show that f is surjective.

Pick any $y \in [0,\infty)$ ~~$\in \mathbb{R}$~~. We will show that there exists $x \in \mathbb{R}$ ~~$y \in [0,\infty)$~~ such that $y\ [f(x)] = x^2$. We need to find $x \in \mathbb{R}$ ~~$y \in [0,\infty)$~~ such that $y\ [f(x)] = x^2$.

error

Choose $x = \sqrt{y}$. We have,

~~$f(1) = (1)^2$ which is equivalent to~~

~~$f(1) = 1$.~~

$y = (\sqrt{y})^2$ which is equivalent to

$y = y. \blacksquare$

10. Let $f: \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $f(n) = (2n+1, n+2)$. Check whether this function is injective and whether it is surjective. Prove your answer.

Injective.

Pick any $x, y \in \mathbb{Z}$. We want to show that

$$(2x+1, x+2) = (2y+1, y+2) \Rightarrow x = y. \quad (1)$$

~~Assume LHS of (1) is true we have~~

$$\cancel{2x+1 = 2y+1 \text{ and } x+2 = y+2}. \quad (2)$$

~~For $2x+1 = 2y+1$, by simplifying we will get~~

$$\cancel{2x = 2y}$$

~~By simplifying (2), we get~~

$$\cancel{x = y}$$

Assume LHS of (1) is true. We have

$$x+2 = y+2 \text{, which implies}$$

$$x = y. \blacksquare$$

7. $f: \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{2\}$,

$$f(x) = \frac{2x+1}{x-1}$$

is bijective.

- Check if it is injective.

Pick any $x, y \in \mathbb{R} \setminus \{1\}$ ~~[and y ∈ ℝ∖{2}]~~

We want to show that

$$\frac{2x+1}{x-1} = \frac{2y+1}{y-1} \Rightarrow x = y \quad (1).$$

Assume ↑ ↑. We want to show that $x=y$.

By multiplying each side by the product of their denominators of (1), we have get

$$(2x+1)(y-1) = (2y+1)(x-1), \text{ which is equivalent to}$$

$$2xy + y - 2x - 1 = 2yx + x - 2y - 1. \quad (2)$$

By simplifying (2), we get

$$3y = 3x \text{ which implies}$$

$$x = y. \ \blacksquare$$

Therefore, ~~it is injective~~ the function is injective.

- Check if ~~it~~ is surjective.

Pick any $y \in \mathbb{R} \setminus \{2\}$. We want to show that

$$\exists x \in \mathbb{R} \setminus \{1\} \text{ such that } \frac{2x+1}{x-1} = y.$$

We will find $x \in \mathbb{R} \setminus \{1\}$ such that $\frac{2x+1}{x-1} = y$. (☆)

Pick $x = 2$. We get

$$\frac{2(2)+1}{2-1} = y, \text{ which is equivalent to}$$

$$5 = y.$$

Since $5 \in \mathbb{R} \setminus \{2\}$, we have shown that the function is surjective.

Pick $x = \frac{-y-1}{2-y}$. We ~~(get)~~ have

Equal if expression

Equivalent if there's equal.

$$f(x) = \frac{2\left(\frac{-y-1}{2-y}\right) + 1}{\frac{-y-1}{2-y} - 1} \quad (1)$$

The RHS of (1) can be simplified to:

$$\frac{\frac{-2y-2}{2-y} + \frac{2-y}{2-y}}{\frac{-y-1}{2-y} - \frac{2-y}{2-y}}$$ which is equal ~~equivalent~~ to,

$$\frac{\frac{-2y-2+2-y}{2-y}}{\frac{-y-1-2+y}{2-y}}$$ which is equal ~~equivalent~~ to

$$\frac{-3y}{-3}$$ which is equal ~~equivalent~~ to

y.

Since $f(x) = y$, we have shown that the function is surjective.

Therefore, the function is bijective.

6.) Consider the following relation on the set $A = \{n \in \mathbb{N} : n \geq 2\}$.

$$aRb \iff \gcd(a,b) = 1$$

- Check if reflexive.

Because $\gcd(3,3) = 3$, we have $3 \not R 3$.

So, R is not reflexive.

- Check if symmetric.

Pick, $a,b \in A$. We want to show that

any $\forall a,b \in A$, $aRb \Rightarrow bRa$.

Assume aRb. We need to show bRa.

Because aRb, we have

$$\gcd(a,b) = 1.$$

We know that

$$\gcd(b,a) = \gcd(a,b) = 1.$$

Therefore, $aRb \Rightarrow bRa$.

- Check if transitive.

Because $\gcd(5,9) = 1$ and $\gcd(9,10) = 1$, but $\gcd(5,10) = 5 \neq 1$,

we have ~~$5 \not R 10$~~ $5R9$ and $9R10$ but $5 \not R 10$.

Therefore, R is not transitive.

- Check if total.

Because $3 \not R 6$ and $6 \not R 3$, R is not total.

- Check if asymmetric.

Because $2R3$ and $3R2$ but $2 \neq 3$, R is not asymmetric.

- Check if dense.