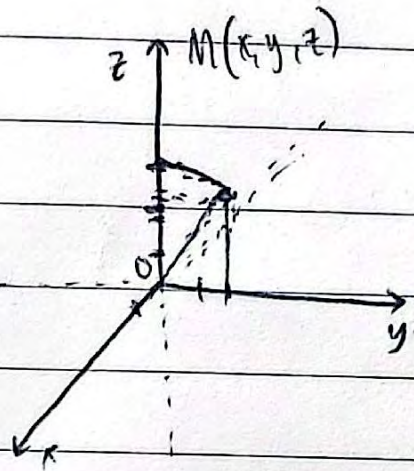
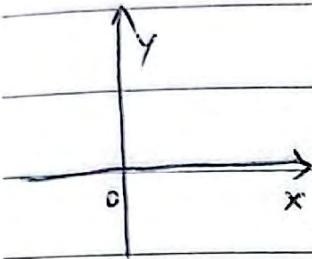


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Math 314 (calc 3)

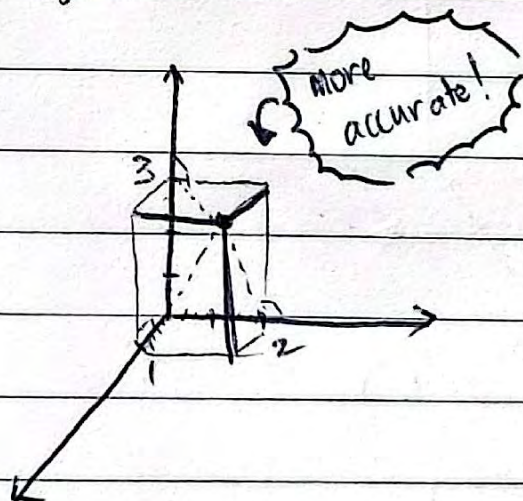
coordinate - system:



Locate the point ~~(1, 2, 3)~~

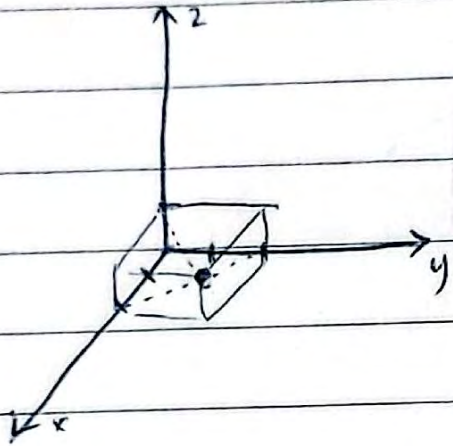
$(1, 2, 3)$

Again!

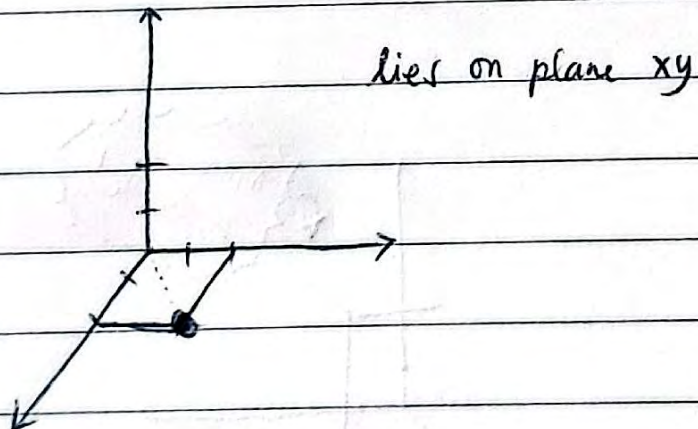


Draw the following points.

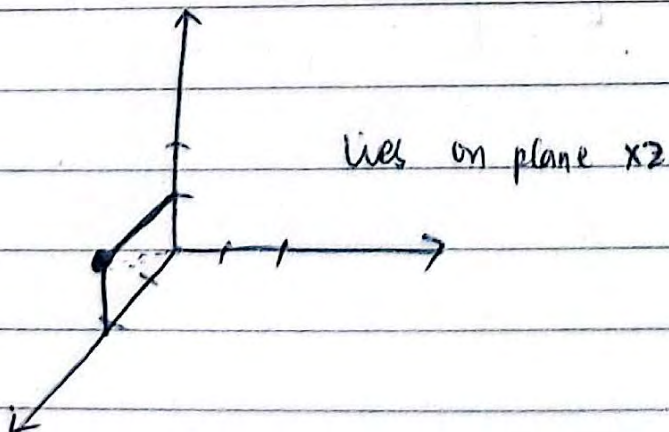
$(2, 2, 1)$



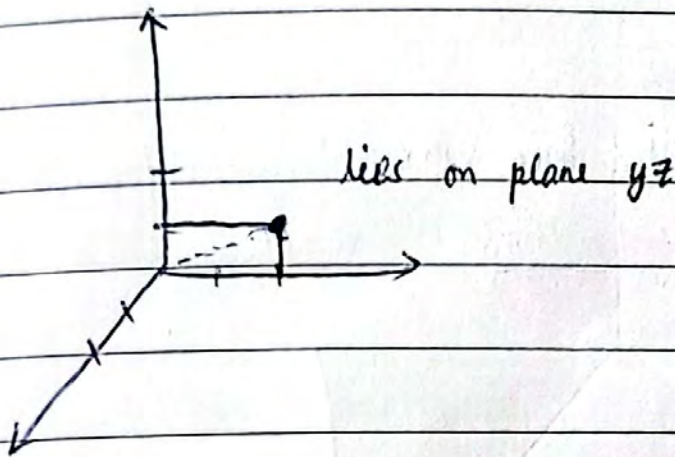
$(2, 2, 0)$



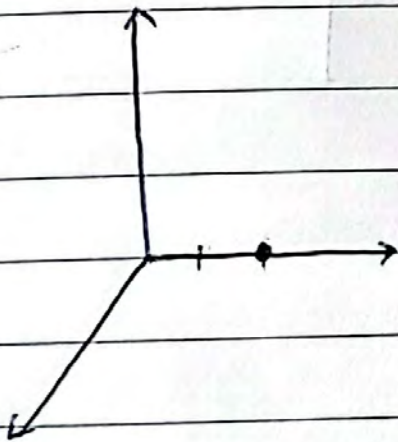
$(2, 0, 1)$



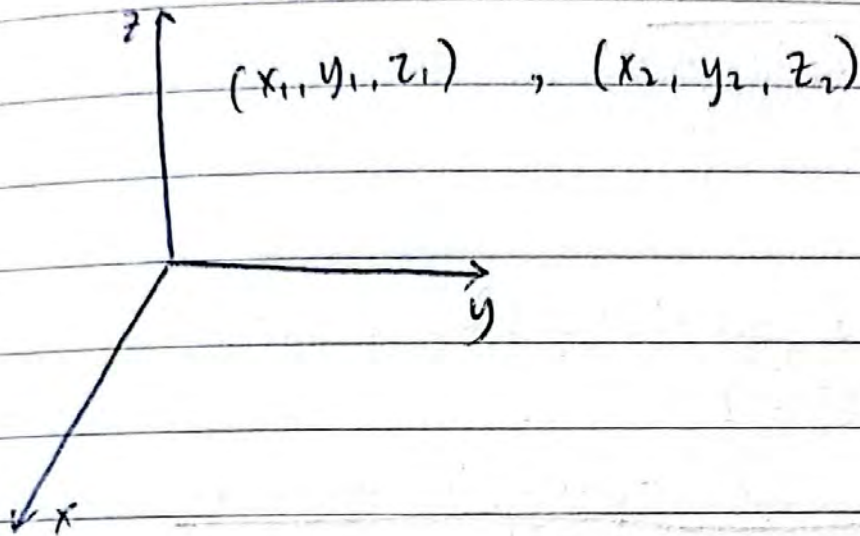
$(0, 2, 1)$



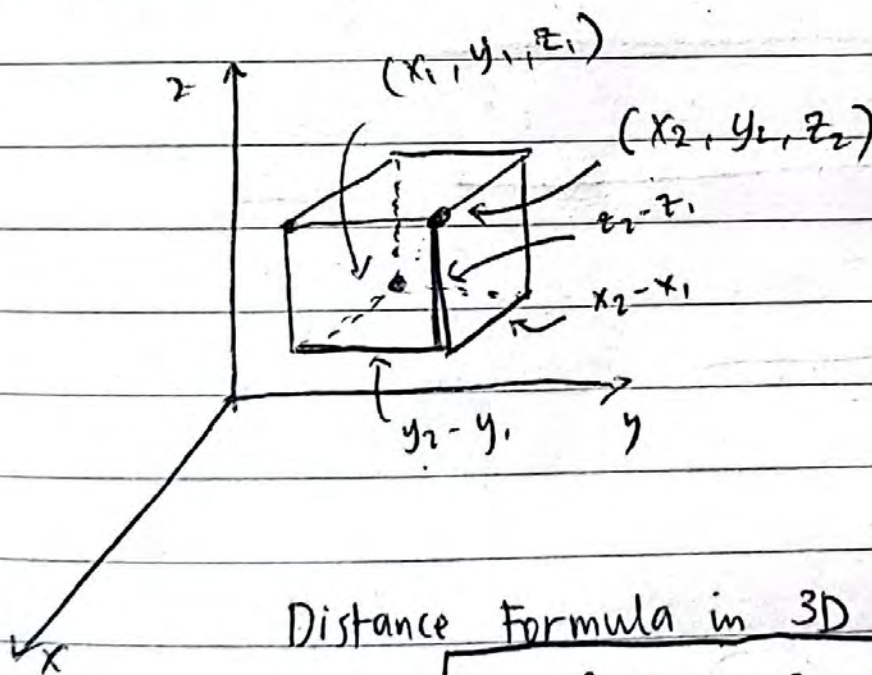
$(0, 2, 0)$



Math 314 (calc 3)



what is the distance between
 (x_1, y_1, z_1) and (x_2, y_2, z_2) ?



Distance Formula in 3D

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Distance of a point to the origin
in 3D

$$OM = \sqrt{a^2 + b^2 + c^2}$$

The set of all points (x, y, z) such that

$$x^2 + y^2 + z^2 = 1$$

is the sphere of radius 1 centered at origin.

The equation of the sphere of radius r
centered at (a, b, c) is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Ex: the eq. of the sphere of radius 2
centered at $(0, -1, 1)$ is

$$x^2 + (y+1)^2 + (z-1)^2 = 4$$

Is the point $(1, 0, 0)$ on the sphere?

plug in $(1, 0, 0)$ to x, y, z

Use the distance formula to know if it is inside or outside of sphere.

Ex: Is the equation

$$x^2 + y^2 + z^2 - 3x + 4y - z = -1$$

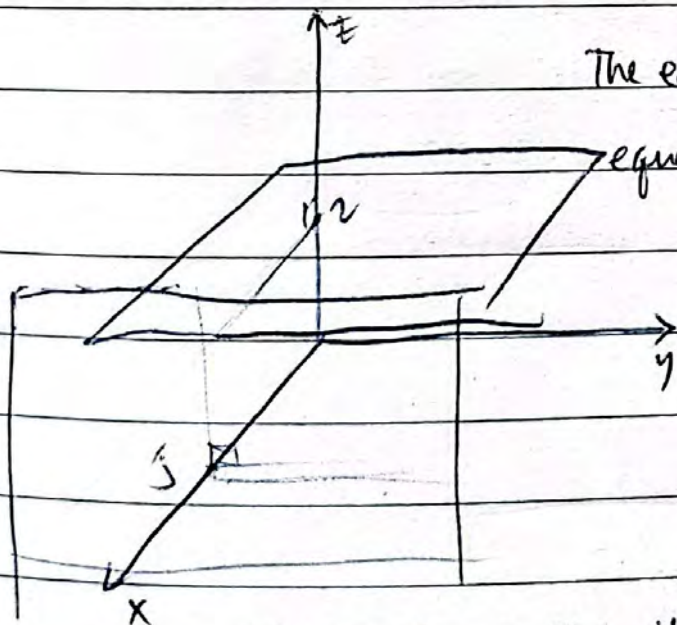
an equation of a sphere?

$$(x^2 - 3x + \frac{9}{4})^2 + (y^2 + 4y + 4)^2 + (z^2 - z + \frac{1}{4})^2 = -1 + \frac{9}{4} + 4 + \frac{1}{4}$$

$$(x - \frac{3}{2})^2 + (y + 2)^2 + (z - \frac{1}{2})^2 = \frac{11}{2}$$

$(a, b, c) = (\frac{3}{2}, -2, \frac{1}{2})$: center

$$r = \sqrt{\frac{11}{2}}$$



The equation $z=2$ is the equation of a plane.

Is $(1, 1, 1)$ on this plane?

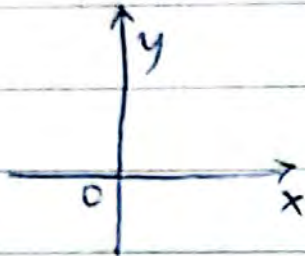
No, it is under the plane.

The equation $x=5$ is the equation of a plane.

What is the equation of the plane xz ? $y=0$

Equation of the plane xy is $z=0$.

Equation of the plane yz is $x=0$.



If the rotation from x -ray to y -ray is counterclockwise then we say that the coordinate system is positively oriented.

Otherwise, it is negatively oriented.

Right-hand Rule: fingers curl from Ox to Oy .

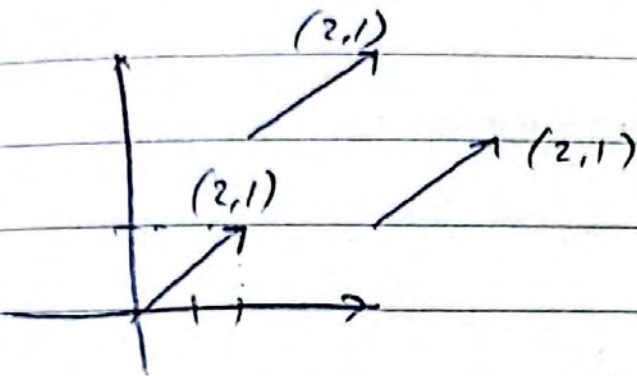
If the thumb points toward Oz : positive orientation. Otherwise, negative orientation.

Vectors

A 2D vector is a pair (a,b) .

A 3D vector is a triple (a,b,c) .

Geometrically, a vector is an arrow.



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Math 314

pair, triple
Vectors $\leftarrow$ arrow w/ certain length and direction

Algebra of vectors

* Addition : $u = (a, b, c)$

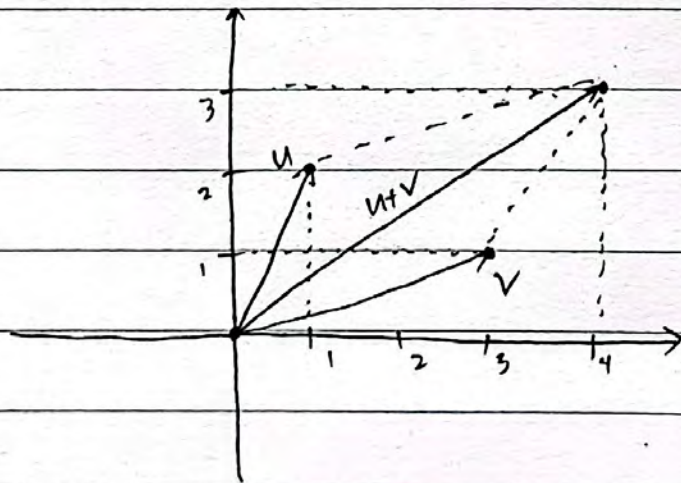
$$v = (d, e, f)$$

$$u+v = (a+d, b+e, c+f)$$

Ex:

$$\left. \begin{array}{l} u = (1, 2) \\ v = (3, 1) \end{array} \right\} u+v = (4, 3)$$

Draw $u, v, u+v$

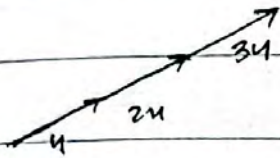


An equivalent way of adding two vectors is to use
the parallelogram rule.

• Scaling :

$$u = (a, b, c)$$

$$ku = (ka, kb, kc)$$

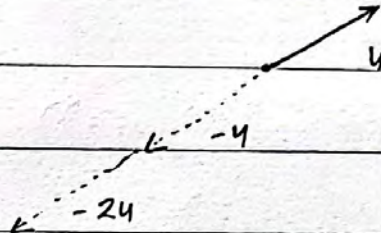


$$u = (1, 2, \text{---})$$

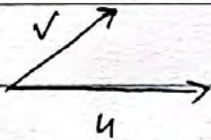
$$2u = (2, 4)$$

$$3u = (3, 6)$$

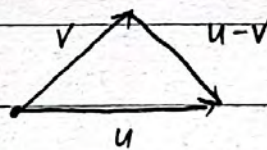
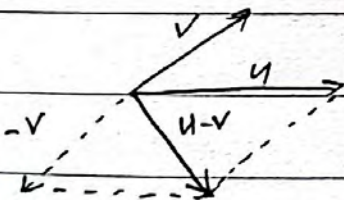
$$-u = (-1)u = (-1, -2)$$



Subtraction :



$$u - v = u + (-v)$$



Ex: $u = (1, 2, 3)$

$v = (-1, 0, 1)$

* The book uses $\langle \dots \rangle$ for vectors,
and $(\dots)$ for points.

$u = \langle 1, 3 \rangle$, $M(1, 3)$

Find $2u + 3v$

$2u = (2(1), 2(2), 2(3))$

$2u = (2, 4, 6)$

$3v = (3(-1), 0(3), 1(3))$

$3v = (-3, 0, 3)$

$2u + 3v = (2 + (-3), 4 + 0, 6 + 3)$

$= (-1, 4, 9)$

In the book:

$u = (1, 2) = 1 \cdot \underset{(1,0)}{i} + 2 \cdot \underset{(0,2)}{j}$

$u = (-2, 3, 1)$

$= -2i + 3j + 1k$

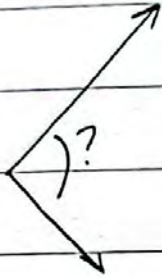
$i = (1, 0, 0)$ unit vector on x -axis

$j = (0, 1, 0)$ " " " y -axis

$k = (0, 0, 1)$ " " " z -axis

Vector Product $\swarrow$ dot product
 $\searrow$ cross product

Dot Product



$$u = (a, b, c)$$

$$v = (d, e, f)$$

$$u \cdot v = v \cdot u$$

$$u \cdot v = (ad) + (be) + (cf)$$

$$u \cdot (2u + v) = u \cdot 2u + u \cdot v$$

$$u = (3, 4)$$

$$v = (2, -1)$$

$$u \cdot v = (3 \cdot 2) + (4 \cdot -1)$$

$$= 6 - 4$$

$$= 2$$

$$u \cdot u = (3 \cdot 3) + (4 \cdot 4)$$

$$= 9 + 16$$

$$= 25$$

$$u \cdot (2u + v) = u \cdot (8, 7)$$

$$2u = (6, 8)$$

$$= 24 + 28$$

$$v = (2, -1)$$

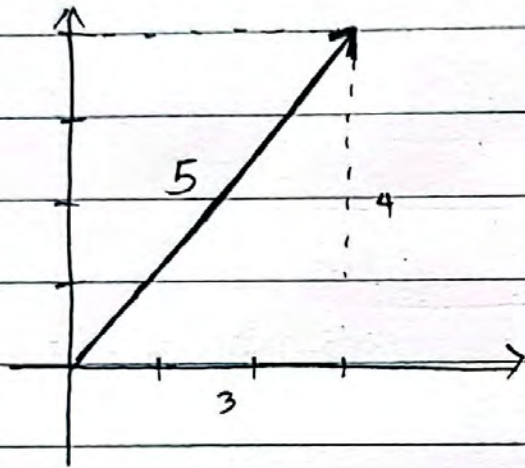
$$= 52$$

$$2u + v = (8, 7)$$

Length of a Vector

Ex: $u = (3, 4)$

Draw u and find its length



$$|u| = \sqrt{a^2 + b^2}$$

length of $u = (a, b, c)$ is

$$|u| = \sqrt{a^2 + b^2 + c^2}$$

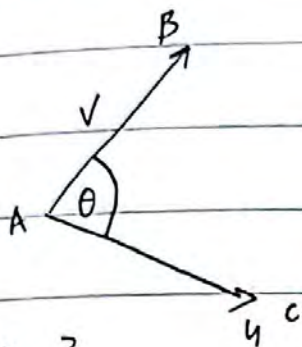
$$|u|^2 = a^2 + b^2 + c^2 = (a \cdot a) + (b \cdot b) + (c \cdot c)$$

$$= (a, b, c) \cdot (a, b, c)$$

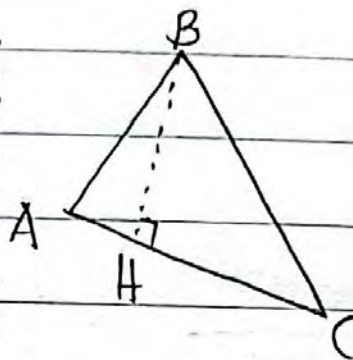
$$= u \cdot u$$

Interesting fact $\div$ $u \cdot u = |u|^2$

Angle between two vectors



$\theta = ?$



$$\cos \theta = \frac{|v|^2 + |u|^2 - |u-v|^2}{2|u||v|}$$

$$= \frac{v \cdot v + u \cdot u - (u-v) \cdot (u-v)}{2|u||v|}$$

$$= \frac{v \cdot v + u \cdot u - (u \cdot u - v \cdot u - u \cdot v + v \cdot v)}{2|u||v|}$$

$$= \frac{2u \cdot v}{2|u||v|} = \frac{u \cdot v}{|u||v|}$$

$$\cos \theta = \frac{u \cdot v}{|u||v|}$$

$$BC^2 = BH^2 + HC^2$$

$$= (\underbrace{AB \sin A}_{BH})^2 + (AC - AH)^2$$

$$= AB^2 \sin^2 A + (AC - \underbrace{AB \cos A}_{AH})^2$$

$$= \boxed{AB^2 \sin^2 A} + AC^2 + \boxed{AB^2 \cos^2 A} - 2AC AB \cos A$$

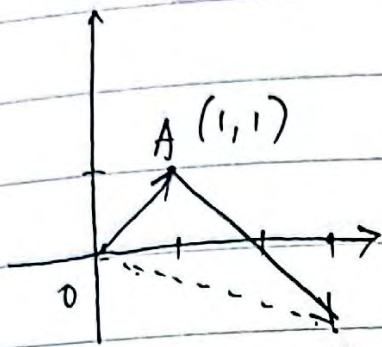
$$= AB^2 (\sin^2 A + \cos^2 A) + AC^2 - 2AC AB \cos A$$

$$= AB^2 + AC^2 - 2AC AB \cos A = BC^2$$

$$\cos A = \frac{AB^2 + AC^2 - BC^2}{2AC AB}$$

Law of Cosine

Math 314



$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$\vec{AB} = (3, -1) - (1, 1)$$

$$\vec{AB} = (2, -2)$$

B (3, -1)

Cross Product

If $a = (a_1, a_2)$ and $b = (b_1, b_2)$ then

$$a \times b = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \quad \leftarrow \text{determinant}$$

If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ then

$$a \times b = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$* a \times b = -b \times a$$

$$= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

Ex: $a = (1, 2, 3)$

$b = (3, 1, -5)$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & -5 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 1 & -5 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 1 & -5 \\ 1 & 2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & -5 \end{vmatrix}$$

$$a \times b = (-10 - 3, 9 + 5, 1 - 6)$$

$$= (-13, 14, -5)$$

$$b \times a = (3 + 10, -5 - 9, 6 - 1)$$

$$= (13, -14, 5)$$

Ex: $a = (1, 3)$

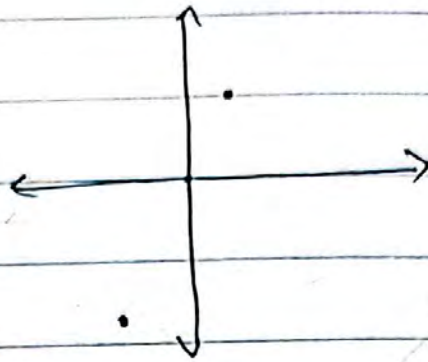
$b = (-2, -5)$

$a \times b, b \times a?$

$a \times b = -5 - (-6)$

$= 1 //$

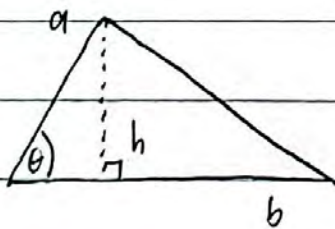
$b \times a = -1 //$



Geometric Nature of Cross Product:

a, b are 3D-vectors.

$a \times b$ is a 3D vector $\left\{ \begin{array}{l} \text{perpendicular to both } a \text{ and } b \\ \text{directed by the right-hand rule} \\ \text{length is (equal) equal to the area} \\ \text{of the parallelogram by } a \text{ and } b, \\ \text{which is } |a||b|\sin\theta \end{array} \right.$



area of triangle $= \frac{1}{2} h |b|$

$= \frac{1}{2} |a| \sin\theta |b|$

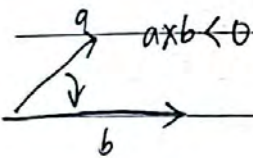
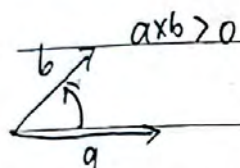
$= \frac{1}{2} |a||b| \sin\theta$

$$|a \times b| = |a| |b| \sin \theta$$

$$a \cdot b = |a| |b| \cos \theta$$

If a and b are 2D vectors then

$a \times b$ is a number $\int$ positive if $a \rightarrow b$ is counterclockwise
absolute value is equal to area of the parallelogram



Normal Vector of the Plane

* Find two vectors then Cross Product

~~chill jollibee echelam for pang nabalacan~~

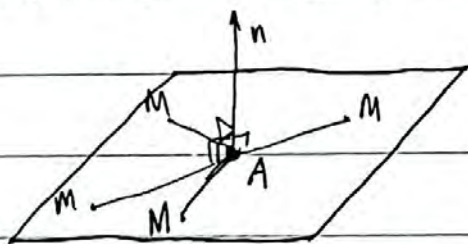
Math 314

A plane is uniquely determined by
a point on the plane and a normal vector.

Suppose a plane contains the point $A(x_0, y_0, z_0)$
and perpendicular to vector $n = (a, b, c)$.

We want to write the equation of this plane.

[Suppose $M(x, y, z)$ lies on this plane.
What equation must x, y, z satisfy?]



M is on the plane if and only if

$\vec{AM}$ is perpendicular to vector n .

M is on the plane if and only if

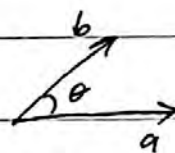
$$\vec{AM} \cdot n = 0$$

$$A(x_0, y_0, z_0)$$

$$M(x, y, z)$$

$$\vec{AM} = (x - x_0, y - y_0, z - z_0)$$

$$n = (a, b, c)$$



$\theta = 90^\circ$ if and only if

$$\cos \theta = 0$$

$$\cos \theta = \frac{a \cdot b}{|a||b|}$$

a is perpendicular to b if
and only if $a \cdot b = 0$

Equation of the plane

$$\vec{AM} \cdot \vec{n} = \boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

Ex: Find the equation of the plane passing through the origin and perpendicular to vector $(1, 2, 3)$

$O(0, 0, 0)$

normal vector is $(1, 2, 3)$

$$1(x-0) + 2(y-0) + 3(z-0) = 0$$

$$x + 2y + 3z = 0$$

A line is determined by a point in the line and a direction vector.

$$\vec{a} = (a_1, a_2, a_3)$$

$A(x_0, y_0, z_0)$

M

We want to write the equation of this line

* M is on the line if and only if $\vec{AM}$ is parallel to vector $\vec{a}$.

* M is on the line if and only if

$$\vec{AM} = t \vec{a} \quad (t \in \mathbb{R})$$

$$(x-x_0, y-y_0, z-z_0) = t(a_1, a_2, a_3)$$

$$= (ta_1, ta_2, ta_3)$$

$$x - x_0 = ta_1$$

$$y - y_0 = ta_2$$

$$z - z_0 = ta_3$$

$$S_0 =$$

$$\boxed{\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}}$$

SYMMETRIC EQUATION

Ex: Find the symmetric equation of the line

passing through $A(1, 2, 1)$ and $B(2, -1, 3)$

direction vector

$$\vec{AB} = (1, -3, 2)$$

Symmetric eq.

$$\frac{x - 1}{1} = \frac{y - 2}{-3} = \frac{z - 1}{2}$$

Parametric equation of the line:

$$\begin{cases} x = x_0 + ta_1 \\ y = y_0 + ta_2 \\ z = z_0 + ta_3 \end{cases} \quad (t \in \mathbb{R})$$

Math 314

Cylinders and quadratic surfaces

A cylinder is a shape given by an equation of x, y, z in which one of the variables is missing.

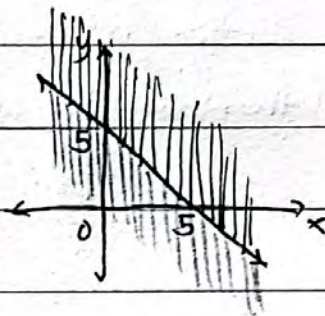
Ex. $x + y = 5$

$$y^2 + z^2 = 1$$

$$x = z^2$$

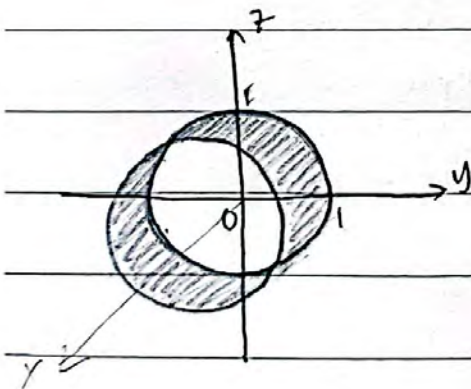
$$z = 5$$

$$x + y = 5$$



Ex: $y^2 + z^2 = 1$

This equation describes the set of all points (x, y, z) such that $x + y = 5$.



quadric

A ~~quadratic~~ surface is a shape given by the equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

(A, B, C, D, E, F are all equal to 0).

Ex: $x^2 + 2y^2 + z^2 - 6 = 0$

$A=1, B=2, C=1, D=E=F=G=H=I=0, J=-6$.

An x -cross section is the intersection of the surface and the plane $x = \text{const.}$

x -cross section of this surface is described by the equation

$$2y^2 + z^2 = \text{const.} \rightarrow \text{This is an ellipse}$$

$$\frac{y^2}{1/2} + \frac{z^2}{1} = 1$$

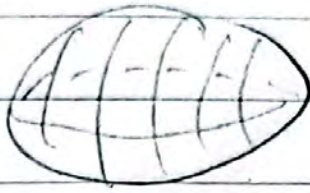
y -cross section of this surface is describe by the equation

$$x^2 + z^2 = \text{const.} \rightarrow \text{This is a circle (ellipse)}$$

z -cross section: $x^2 + 2y^2 = \text{const.} \rightarrow \text{Also an ellipse.}$

ellipse
 ellipse
 ellipse

} ellipsoid



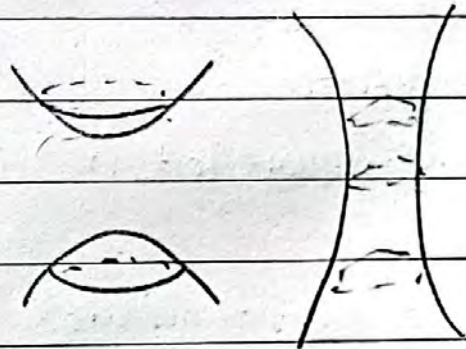
ellipse
 ellipse
 parabola
 parabola

} elliptic paraboloid



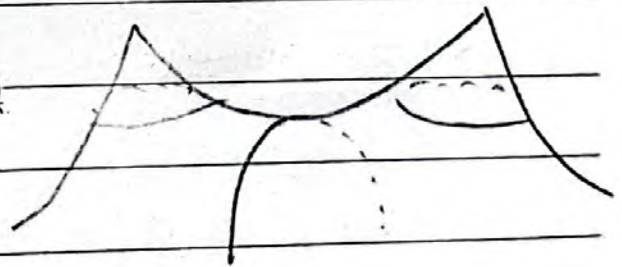
ellipse
 hyperboloid
 hyperbola
 hyperbola

} elliptic hyperboloid



hyperbola
 parabola
 parabola

} hyperbolic paraboloid



Math 314

chapter 13: curves

A curve is a function $r: [a, b] \rightarrow \mathbb{R}^2$ or $\mathbb{R}^3$.

Ex: $r(t) = (\cos t, \sin t)$

✓ This is a parametric equation of a curve $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

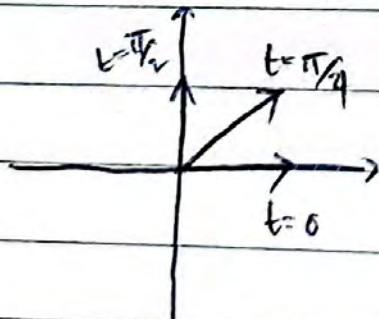
Ex: $r(t) = (\cos(t), \sin(t), t)$

✓ This is a parametric equation of a curve in 3D space:

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

A vector function is a function whose values are vectors.

Ex: $r(t) = (\cos t, \sin t)$



Vector function $\leftrightarrow$ function

curve $\leftrightarrow$ graph

We'll do Calculus on Vector functions.

{ take limit
derivative
integral

$$\lim_{x \rightarrow 1} \overset{\text{vector}}{(e^x, \sin x, \ln x)} = (e, \sin(1), 0)$$

* to get the limit of a vector, you take the limit of each component and then put together as a vector.

$$\text{ex } \lim_{x \rightarrow 0} \left(\frac{1}{x}, e^x, x \right) = \text{DNE}$$

* if one of the component does not exist, then the whole vector DNE.

Math 314

$\lim_{t \rightarrow t_0} \mathbf{r}(t)$
 $\mathbf{r}(t)$ is a
 vector
 function

Derivative of a vector function

$$\mathbf{r}(t) = (t^2, \frac{1}{t}, e^t)$$

$$\mathbf{r}'(t) = (2t, -\frac{1}{t^2}, e^t)$$

To take the derivative of a vector function,
 you take the derivative of each component
 and put together as a vector.

Alternative Definition:

$$\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \quad \text{* Why?}$$

Write $\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t))$

$$\mathbf{r}(t+h) = (r_1(t+h), r_2(t+h), r_3(t+h))$$

$$r(t+h) = (r_1(t+h), r_2(t+h), r_3(t+h))$$

$$r(t) = (r_1(t), r_2(t), r_3(t))$$

$$\frac{r(t+h) - r(t)}{h} = \left(\frac{r_1(t+h) - r_1(t)}{h}, \dots, \dots \right)$$

$$= \left(\frac{r_1(t+h) - r_1(t)}{h}, \dots, \dots \right)$$

$$\lim_{h \rightarrow 0} \frac{r(t+h) - r(t)}{h} = \left(\lim_{h \rightarrow 0} \frac{r_1(t+h) - r_1(t)}{h}, \dots, \dots \right)$$

$$r'(t) = (r'_1(t), r'_2(t), r'_3(t))$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Ex: Find the derivative of

$$r(t) = (t, \sqrt{t}, \ln(t+1))$$

$$r'(t) = \left(1, \frac{1}{2\sqrt{t}}, \frac{1}{t+1} \right)$$

$$r(t)$$

Ex: Find all the values of t where vector function

$$r(t) = (t, \sqrt{t}, |t-1|)$$

is differentiable. Then find $r'(t)$.

$$r_1(t) = t \quad = r'_1(t) = 1$$

$$r_2(t) = \sqrt{t} \quad = r'_2(t) = \frac{1}{2\sqrt{t}}$$

$$r_3(t) = |t-1| \quad \begin{matrix} 1 \\ -1 \end{matrix}$$

$$\lim_{h \rightarrow 0} \frac{|(t+h)-1| - |t-1|}{h}$$

$r(t)$ is differentiable if t , $\sqrt{t}$, $|t-1|$ are differentiable.

Function t is diff. for all t

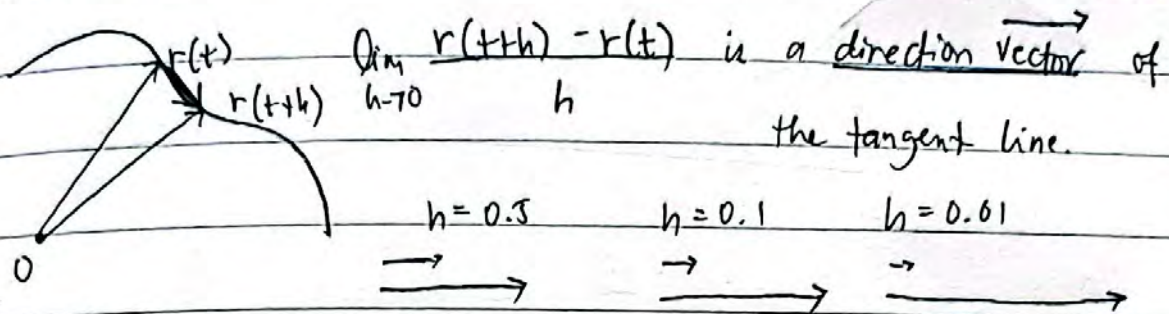
Function $\sqrt{t}$ is differentiable for all $t > 0$.

Function $|t-1|$ is differentiable for all $t \neq 1$.

Therefore, $r(t)$ is differentiable for $t > 0$, $t \neq 1$.

$$r'(t) = \begin{cases} (1, \frac{1}{2\sqrt{t}}, 1) & \text{if } t > 1 \quad t-1 > 0 \\ (1, \frac{1}{2\sqrt{t}}, -1) & \text{if } t < 1 \quad -(t-1) > 0 \end{cases}$$

Geometric meaning of derivative

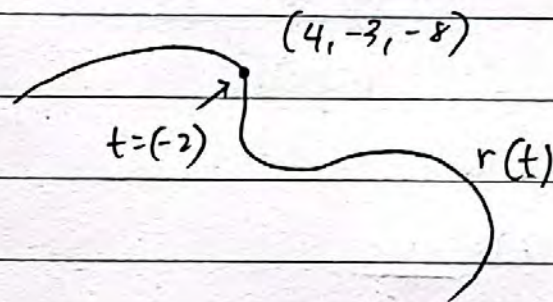


Ex. Find the equation of the tangent line to the curve $r(t) = (t^2, t-1, t^3)$ at the point $(4, -3, -8)$

$$r(t) = (t^2, t-1, t^3)$$

$$r'(t) = (2t, 1, 3t^2)$$

$$r'(-2) = (-4, 1, 12)$$



Symmetric Equation

$$\frac{x-4}{-4} = \frac{y+3}{1} = \frac{z+8}{12}$$

Parametric Equation

$$\begin{cases} x = 4 + (-4)t \\ y = -3 + t \\ z = -8 + 12t \end{cases} \quad (t \in \mathbb{R})$$

Length of the curve $r(t)$, $a \leq t \leq b$



$$L = \int_a^b |r'(t)| dt$$

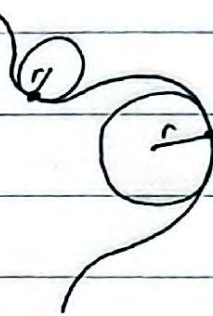
length

Curvature of the curve $r(t)$ is

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

length

The smaller
the circle,
the higher
the curvature.



$$k(t) = \frac{1}{R(t)}$$

Math 314

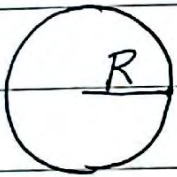
Curvature of a curveNote: curvature is a number.

: measures how bent the curve is at a given position.

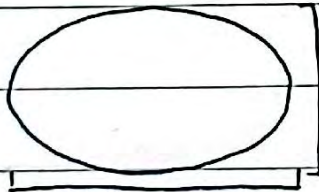
$$k = \frac{1}{R}$$

$R(t) \rightarrow$ is the radius of the circle that best fits the curve at position t .

Ex: A straight line has constant curvature, which is 0.



A circle of radius R has a constant curvature, which is $k = \frac{1}{R}$.

Ex:

max. curvature

min. curvature

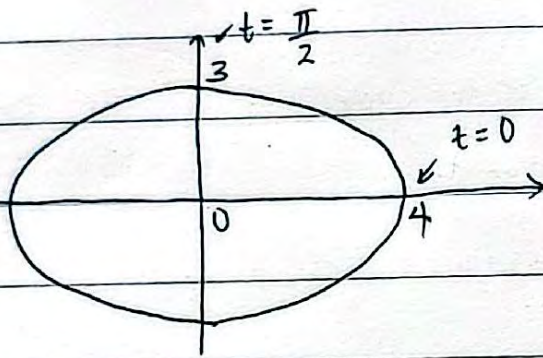
Formula of curvature:

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Ex: Consider an ellipse

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$

Find the positions on the curve where the curvature is maximum and minimum.



$$\mathbf{r}(t) = (\underbrace{4 \cos t}_x, \underbrace{3 \sin t}_y), \quad 0 \leq t \leq 2\pi$$

why?

↓

$$\frac{(4 \cos t)^2}{4^2} + \frac{(3 \sin t)^2}{3^2} =$$

$$\cos^2(t) + \sin^2(t) = 1$$

compute $k(t)$ using the formula of curvature.

$$\mathbf{r}'(t) = (-4 \sin t, 3 \cos t)$$

$$\mathbf{r}''(t) = (-4 \cos t, -3 \sin t)$$

$$|r'(t) \times r''(t)| = \begin{vmatrix} -4 \sin t & 3 \cos t \\ -4 \cos t & -3 \sin t \end{vmatrix}$$

$$12 \sin^2 t - (-12 \cos^2 t)$$

$$12 \sin^2 t + 12 \cos^2 t$$

$$12(\sin^2 t + \cos^2 t)$$

$$12$$

$$\begin{aligned} |r'(t)|^3 &= \sqrt{(-4 \sin t)^2 + (3 \cos t)^2} \\ &= \sqrt{16 \sin^2 t + 9 \cos^2 t} \\ &= \sqrt{4^2(\sin^2 t) + 3^2(\cos^2 t)} \end{aligned}$$

$$12$$

$$= \frac{12}{(4^2 \sin^2 t + 3^2 \cos^2 t)^{3/2}}$$

$$t=0 = \frac{12}{(\sqrt{16(0) + 9(1)})^3} = \frac{12}{(\sqrt{9})^3} = \frac{12}{27} = \frac{4}{9}$$

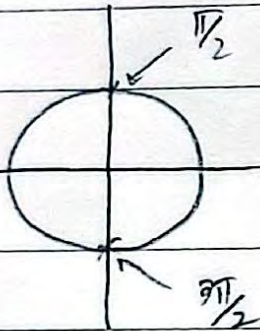
$$t=\pi/2 = \frac{12}{(\sqrt{16(1) + 9(0)})^3} = \frac{12}{(\sqrt{16})^3} = \frac{12}{64} = \frac{3}{16}$$

Formula of the curvature:

$$k(t) = \frac{12}{(7 \sin^2 t + 9)^{3/2}}$$

$$k(t) = \frac{12}{(7 \sin^2 t + 9)^{3/2}}$$

$$\sin^2 t = 1 \quad \text{when} \quad \sin t = \pm 1$$



$$t = \frac{\pi}{2} : r\left(\frac{\pi}{2}\right) = (4 \cos \frac{\pi}{2}, 3 \sin \frac{\pi}{2}) = (0, 3)$$

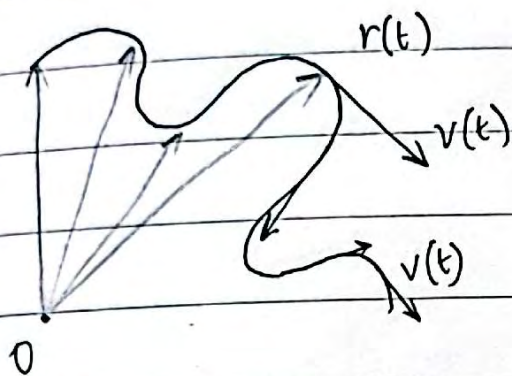
$$t = \frac{3\pi}{2} : r\left(\frac{3\pi}{2}\right) = (0, -3)$$

At $(0, \pm 3)$ the curvature is [minimum]

$$\frac{12}{(7+9)^{3/2}} = \frac{12}{64} = \left(\frac{3}{16}\right) \quad r = \frac{16}{3}$$

At $(\pm 4, 0)$ the curvature is [maximum]

$$\frac{12}{(9)^{3/2}} = \frac{12}{27} = \left(\frac{4}{9}\right) \quad r = \frac{9}{4}$$



$r(t)$: position vector

$$v(t) \text{ def } \lim_{h \rightarrow 0} \frac{r(t+h) - r(t)}{h} = r'(t) : \text{velocity vector}$$

$$\text{speed} = |v(t)| = |r'(t)|$$

$a(t) = v'(t)$ is the acceleration vector.

Chapter 14: Partial derivatives

A multivariable function is a function that has two or more variables.

Ex: $f(x, y) = x + y^2$

$f(x, y, z) = xyz + 5$

$f(x, y) = x^2 + 1 \rightarrow$ a special case, can also be viewed as single variable

Multivariable function $\leftarrow$ domain
graph
level sets.

The domain of $f(x, y)$ is the set of (x, y) such that $f(x, y)$ is well-defined.

Ex: $f(x, y) = x + y$.

The domain of f is $\mathbb{R}^2$ $\leftarrow$ the entire 2D plane.

$f(x, y) = \sqrt{x - y^2}$

The domain of f is $\{(x, y) : x \geq y^2\}$

Ex:

$$f(x, y, z) = \sqrt{3 - x^2 - y^2 - z^2}$$

The domain of f is $\{(x, y, z) : 3 - x^2 - y^2 - z^2 \geq 0\}$ or
 $\{(x, y, z) : x^2 + y^2 + z^2 \leq 3\}$



ball centered at $O(0,0,0)$

with radius $\sqrt{3}$.

Ex: $f(x, y, z) = \sqrt{3 - x^2 - y^2 - 2z^2}$

$$\text{Domain} = \{(x, y, z) : x^2 + y^2 + z^2 \leq 3\}$$

ellipsoid.

Graph of function $f(x, y)$ is the set

$$\{(x, y, z) : z = f(x, y)\}$$

$f(x)$

Graph of f is $\{(x, y) : y = f(x)\}$

Graph of $f(x, y, z)$ is the set

$$\{(x, y, z, a) : a = f(x, y, z)\}$$

Ex

(a) $f(x) = x^2$

parabola

(b) $f(x, y) = x^2 + y^2$

(circular cylinder)

elliptic

(c) $f(x, y) = x^2 - y^2$

(parabolic cylinder)

paraboloid

hyperbolic

paraboloid

(d) $f(x, y) = z^2$

(parabolic) cylinder x

↑ not a valid function. NO GRAPH

A c-level set of function $f(x_1, \dots, x_n)$ is the set $\{(x_1, \dots, x_n) : f(x_1, \dots, x_n) = c\}$.

Ex: what is the 0-level set of $f(x) = x^2 + 2x$?

$$\{x : f(x) = 0\} = \{0, -2\}$$

What is the (-1)-level set of f ?

$$\{x : f(x) = -1\} = \{-1\}$$

$$x^2 + 2x = -1$$

$$x^2 + 2x + 1 = 0$$

$$(x+1)(x+1) = 0$$

$$x = -1$$

What is the (-2)-level set of f ?

$$\{x : f(x) = -2\} = \{x : x^2 + 2x = -2\}$$

$$a = 1$$

$$= \{x : x^2 + 2x + 2 = 0\} = \{\} \text{ (empty set)}$$

$$b = 2$$

$$= \{\text{no.}\}$$

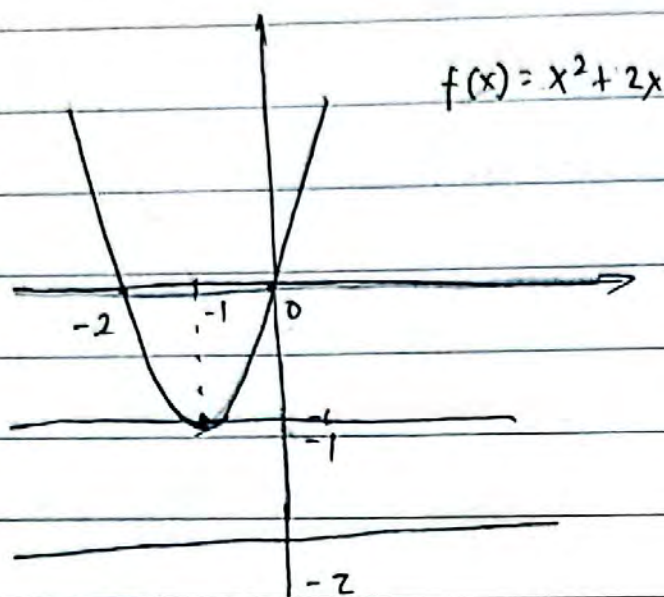
$$c = 2$$

$$\frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2}$$

$$\frac{-2 \pm \sqrt{-4}}{2}$$

$$\frac{-2 \pm 2i}{2}$$

$$-1 \pm i$$

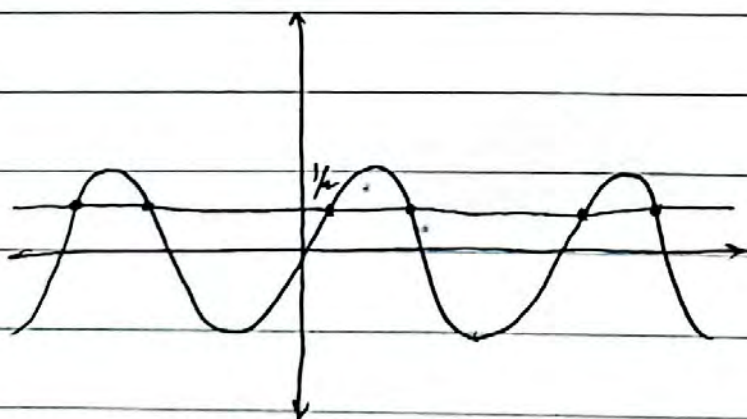


0 - level set is the intersection of the graph of f and the line $y=0$.

-1 - level set intersected once.

$$f(x) = \sin x$$

What is $\frac{1}{2}$ level set?



$\{\infty\}$

infinite empty

* Sine function can only have either $\{\infty\}$ or $\{\}$.

Ex: $f(x,y) = x^2 + 2y^2$

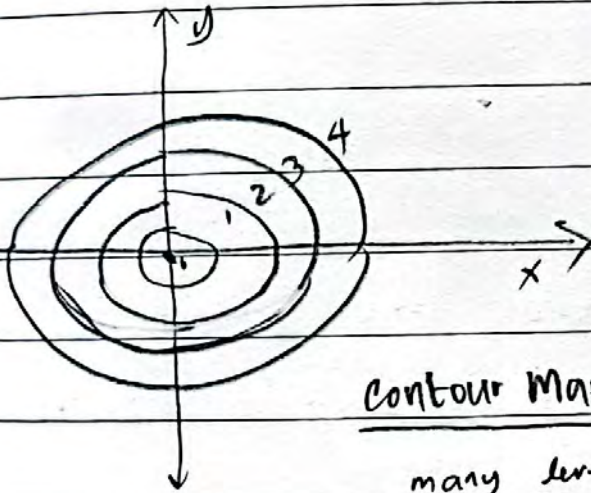
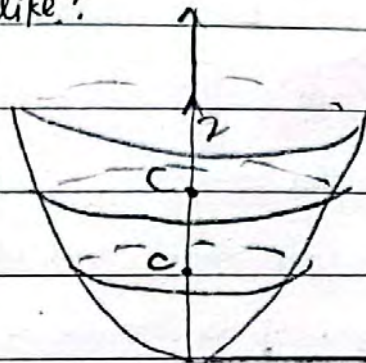
What does 1-level set of f look like?

$$\{(x,y) : f(x,y) = 1\}$$

$$\{(x,y) : x^2 + 2y^2 = 1\} \text{ ellipse}$$

c -level set of f is

$$\{(x,y) : f(x,y) = c\}$$

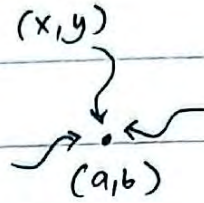


Contour Map - is a collection of many level sets.

Limit

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

$$(x,y) \rightarrow (a,b)$$



What does this mean?

As (x,y) gets closer and closer to (a,b)
(while staying in the domain of f), $f(x,y)$ gets
closer and closer to L .

Ex: $\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x+y} = ? \quad L$

$$(x,y) \rightarrow (0,0)$$

① : y -axis

$$x=0, y \rightarrow 0$$

$$\frac{x}{x+y} = \frac{0}{0+y} = 0, \text{ so } L=0.$$

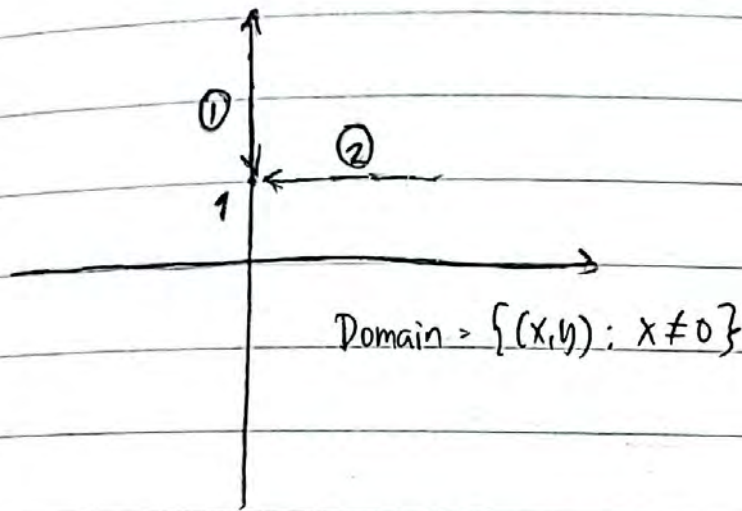
② : x -axis

$$x \rightarrow 0, y=0$$

$$\frac{x}{x+y} = \frac{x}{x} = 1, \text{ so } L=1.$$

limit
doesn't
exist

Ex: $\lim_{(x,y) \rightarrow (1,2)} \frac{e^x - y}{x} = \underline{\underline{e - 2}}$



Ex: $\lim_{(x,y) \rightarrow (0,1)} \frac{e^x - y}{x}$

* To show that $\lim_{(x,y) \rightarrow (0,1)} f(x,y)$ does not exist, we need to find

Try: ① y-axis

two paths inside the domain

$x = 0, y \rightarrow 1$

leading to (x_0, y_0) such that

$\frac{e^0 - 1}{0} = \text{DNE}$
undefined
not in the domain

the function f has different limits on these paths.

② x-axis

$x \rightarrow 0, y = 1$

$\frac{e^0 - 1}{0} = \text{DNE}$

Let's rewrite: (at the back)

On path (1) : $x=0, y \rightarrow 1$

$$\frac{e^x - y}{x} \text{ is undefined}$$

path (1) is not in the domain.

On path (2) : $y=1, x \rightarrow 0$

$$\frac{e^x - y}{x} = \frac{e^x - 1}{x}$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} \xrightarrow{\text{L'Hopital}} \lim_{x \rightarrow 0} \frac{e^x}{1} = e^0 = 1$$

$$\lim_{(x,y) \rightarrow (0,1)} f(x,y) = 1$$

on path 2.

path (3) $y \rightarrow 1, x \rightarrow 0$

eq. of the line $y = x+1$

$$f(x,y) = \frac{e^x - y}{x} = \frac{e^x - (x+1)}{x} = \frac{e^x - x + 1}{x}$$

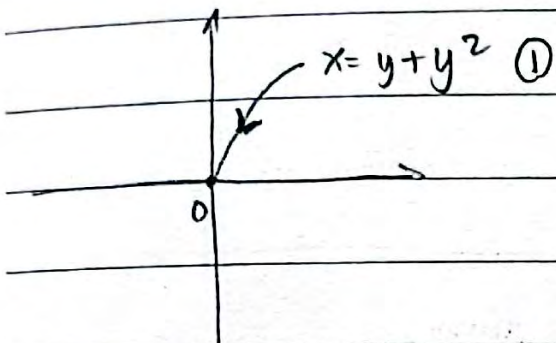
$$\lim_{x \rightarrow 0} \frac{e^x - x + 1}{x} \xrightarrow{\text{L'Hopital}} \lim_{x \rightarrow 0} \frac{e^x - 1}{1} = \frac{e^0 - 1}{1} = \frac{1-1}{1} = \frac{0}{1} = 0$$

$$f(x,y) = 0$$

$(x,y) \rightarrow (0,1)$

$\therefore$ Since path 2 and path 3 have different L, the limit DNE.

Ex: $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{x-y}$



On path ①: $f(x,y) = \frac{y}{x-y} = \frac{y}{y+y^2-y} = \frac{y}{y^2} = \frac{1}{y}$

$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{y \rightarrow 0} \frac{1}{y}$ DNE.

on path 1

Therefore, $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ DNE.

Alternatively, if you can find one path which is in the domain that is undefined, we can conclude that the limit DNE.

Ex: $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^2}$

$$\left| \frac{xy^2}{x^2+y^2} \right| = \frac{|x|y^2}{x^2+y^2} \leq |x|$$

≤ 1

$-|x| \leq \frac{xy^2}{x^2+y^2} \leq |x|$ Squeeze them.

$$\begin{array}{c} \swarrow \quad \downarrow \quad \searrow \\ 0 \end{array}$$

Squeeze Theorem

* Inequality: $|a| \leq b \iff -b \leq a \leq b$

Ex:

$$|x| \leq 1 \iff -1 \leq x \leq 1$$

$$|x| \leq |y| \iff -|y| \leq x \leq |y|$$

More example:

$$\lim_{(x,y) \rightarrow (0,0)} xy \sin\left(\frac{1}{y}\right)$$

① Take the absolute value

$$\underbrace{\left| xy \sin\left(\frac{1}{y}\right) \right|}_a = |x| |y| \left| \sin\left(\frac{1}{y}\right) \right|$$

$$\underbrace{\left| \sin\left(\frac{1}{y}\right) \right|}_{\leq 1} \leq \overbrace{|x||y|}^b$$

$$-|x||y| \leq xy \sin\left(\frac{1}{y}\right) \leq |x||y|$$

as $(x,y) \rightarrow (0,0)$ as $(x,y) \rightarrow (0,0)$

↓
0

Therefore, $\lim_{(x,y) \rightarrow (0,0)} xy \sin\left(\frac{1}{y}\right) = 0.$

Ex: (a) $\lim_{(x,y) \rightarrow (1,0)} \frac{e^x - e^y}{x - y} = \frac{e^1 - e^0}{1 - 0} = \frac{e - 1}{1} = e - 1$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{e^x + e^y}{x - y} = \text{DNE}$

(c) $\lim_{(x,y) \rightarrow (1,1)} \frac{e^x - e^y}{x - y} = \text{Solution at the back}$

$f(x) = e^x$ Mean Value Theorem

$$\frac{e^x - e^y}{x - y} = \frac{f(x) - f(y)}{x - y} = f'(c)$$

where c is some number
between x and y .

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

$$f'(c) = e^c$$

$$x < c < y$$


$$\lim_{(x,y) \rightarrow (1,1)} \frac{e^x - e^y}{x - y} = e$$

Function $f(x,y)$ is said to be continuous at (x_0, y_0)

$$\text{if } \lim_{(x,y) \rightarrow (x_0, y_0)} f(x,y) = f(x_0, y_0)$$

If f is continuous at all $(x_0, y_0) \in A$ then we say
that f is continuous on A .

Ex: The function $f(x,y) = \frac{e^x - e^y}{x - y}$

is continuous on $\{(x,y) : x \neq y\}$

5/16/25

Partial Derivatives

$$f(x,y) = x + yx^2$$

"Freeze" y , then differentiate with respect to x .

$$1 + 2yx$$

$f_x(x,y) = 1 + 2yx$: Partial Derivative of
 f w.r.t. x .

"Freeze" x , then differentiate with respect to y .

$$f_y(x,y) = 0 + x^2 = x^2$$

Partial Derivative of f w.r.t. y .

Ex: $f(x,y) = \frac{x}{x + 2y}$

Find f_x and f_y .

$$f(x,y) = \frac{x}{x+2y}$$

$$(x+2y)(x+2y)$$

$$u = x$$

$$u^2 =$$

$$x^2 + 2yx + 2yx + 4y^2$$

$$v = x + 2y$$

"Freeze" x.

wrong formula
75

$$f_x(x,y) = \frac{x(1) - 1(x+2y)}{(x+2y)^2} \rightsquigarrow \frac{(x+2y) - x}{(x+2y)^2}$$

$$= \frac{x - (x+2y)}{(x+2y)^2} = \frac{-2y}{(x+2y)^2}$$

$$= \frac{-2y}{(x+2y)^2}$$

"Freeze" y.

wrong formula
"

$$f_y(x,y) = \frac{x(2) - (x+2y)(0)}{(x+2y)^2} \rightsquigarrow \frac{(x+2y)(0) - x(2)}{(x+2y)^2}$$

$$= \frac{2x}{(x+2y)^2} \rightsquigarrow \frac{-2x}{(x+2y)^2}$$

$$* \frac{5}{y+1} = 5(y+1)^{-1}$$

$$= 5(-1)(y+1)^{-2}(1)$$

Ex: $f(x, y, z) = \frac{xe^z}{y+z}$

Find f_x, f_y, f_z .

$$f_x(x, y, z) = \frac{xe^z}{y+z}$$

$$= \frac{e^z(y+z) - xe^z(0)}{(y+z)^2}$$

$$= \frac{e^z}{y+z}$$

$$f_y(x, y, z) = \frac{(0)(y+z) - xe^z(1)}{(y+z)^2}$$

$$= \frac{-xe^z}{(y+z)^2}$$

~~$$f_z(x, y, z) = \frac{(0)(y+z) - xe^z(1)}{(y+z)^2}$$~~

$$f_z(x, y, z) = \frac{xe^z(y+z) - xe^z(1)}{(y+z)^2}$$

$$= \frac{xe^z(y+z-1)}{(y+z)^2}$$

Notation :

$$f(x) \rightarrow f'(x), \frac{df}{dx}, f'$$

$$f(x, y) \rightarrow f_x(x, y), f_x, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

" ∂ ": del

" ∂ ": partial

$$(f_x)_x = f_{xx} = \left(\frac{\partial f}{\partial x} \right)_x = \frac{\partial \left(\frac{\partial f}{\partial x} \right)}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \quad f'' = \frac{d^2 f}{dx^2}$$

$$(f_x)_y = f_{xy} = \left(\frac{\partial f}{\partial x} \right)_y = \frac{\partial \left(\frac{\partial f}{\partial x} \right)}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

~~f~~

$$f_{xxxyxy} = \frac{\partial^5 f}{\partial x^3 \partial y \partial z} = \frac{\partial^5 f}{\partial y \partial x \partial y \partial x \partial x} = \frac{\partial^5 f}{\partial x^3 \partial y^2}$$

Ex: $f(x, y, z) = xy^2e^z$

Find $f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy}$.

* $f_{zy} =$

$$f_x = (\cancel{2e^z}) y^2 e^z$$

$$f_{xy} = (\cancel{2}) 2e^z y$$

* f_{yx}

$$f_y = 2xe^z y$$

$$f_{yx} = 2ye^z$$

* f_{xz}

$$f_x = e^z y^2$$

$$f_{xz} = e^z$$

* f_{zx}

$$f_z = e^z xy^2$$

$$f_{zx} = \bullet e^z y^2$$

* f_{yz}

$$f_y = 2xe^z y$$

$$f_{yz} = e^z$$

* f_{zy}

$$f_z = e^z xy^2$$

$$f_{zy} = \bullet 2e^z xy$$

Clairaut's theorem:

$$f_{xy} = f_{yx}$$

for any "smooth" function f .

$$f_{xyz} \quad \text{vs.} \quad f_{zxy}$$

$$\begin{aligned} f_{xyz} &= ((f_x)_y)_z = ((f_x)_z)_y = f_{xzy} = (f(xz))_{yz} \\ &= (f(xz))_{zy} = f_{(zx)zy} \\ &= f_{zxyz} \end{aligned}$$

5/19/2025

Geometric meaning of partial derivatives

$f_x(a,b) = f_x(1,2)$ is calculated by

differentiating f w.r.t. x

and then plug $(x,y) = (1,2)$.

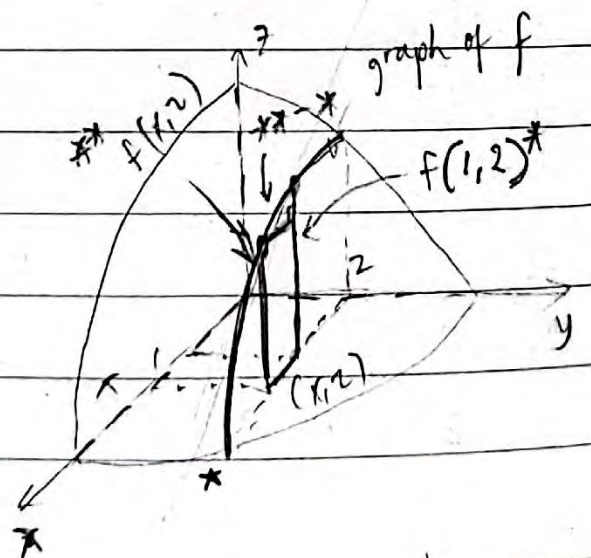
Ex: $f(x,y) = x^3 y^3$

$$f_x(x,y) = 2xy^3$$

$$f_x(1,2) = 2(1)(2)^3 = 16.$$

Alternative way to calculate $f_x(1,2)$

$$f_x(1,2) = \lim_{\lambda \rightarrow 1} \frac{f(\lambda, 2) - f(1, 2)}{\lambda - 1}$$



Parametric Equation of the curve (*)

$$x = t$$

$$y = 2$$

$$z = f(t, 2)$$

The curve (*) is the intersection of the plane $y=2$ and the graph of f .

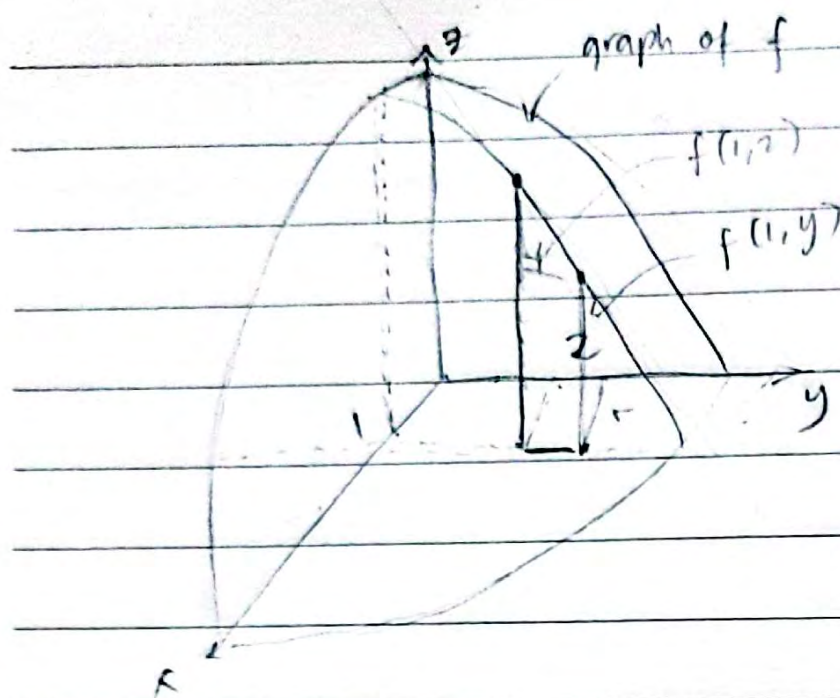
$f'_x(1, 2)$ is the slope of the ~~purple curve~~ tangent line to the curve (*) at the point $(1, 2, f(1, 2))$.

$$f_y(1, 2) = x^3 y^3$$

$$= x^3 2y$$

$$f_y(1, 2) = 1(2)(2) = 5$$

$$f_y(1, 2) = \lim_{y \rightarrow 2} \frac{f(1, y) - f(1, 2)}{y - 2}$$



The ^(**) curve is the intersection of the plane $x=1$ and the graph of f .

$f_y(1,2)$ is the slope of the tangent line to the ^(**) curve at the point $(1,2, f(1,2))$

* Parametric Curve

$$x=1$$

$$y=t$$

$$z = f(1,t)$$

$f_x(1,2) < 0$ — positive negative / 0?

$$f_y(1,2) < 0$$

Function = $f(x,y) = 9 - x^2 - y^2$

Graph is $z = 9 - x^2 - y^2$: elliptic paraboloid

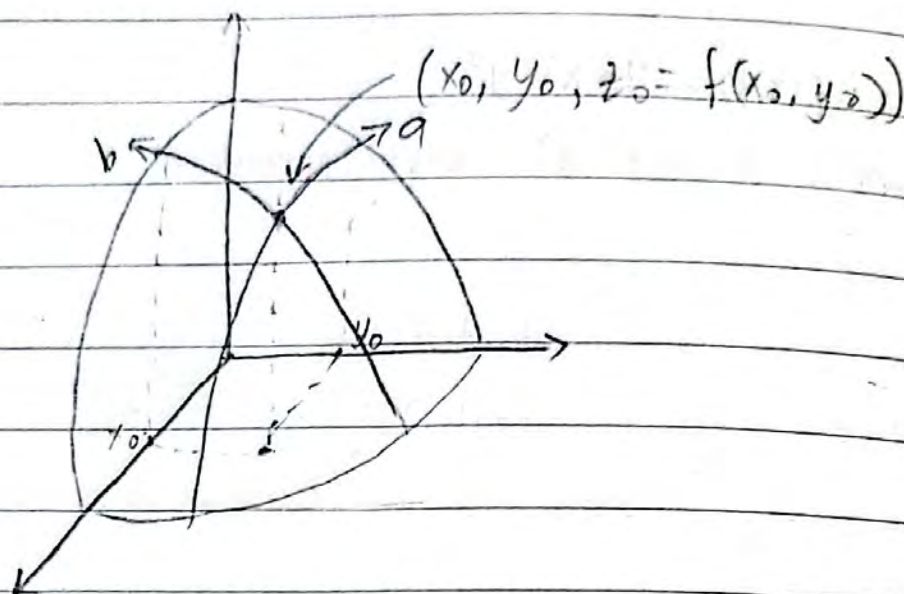


$$f_x(1,2) =$$

$$f_y(1,2) =$$

$$\begin{aligned} f_y(1,2) &= 9 - x^2 - y^2 \\ &= -2y \\ &= -2(2) \\ &= -4 \end{aligned}$$

$$\begin{aligned} f_x(1,2) &= 9 - x^2 - y^2 \\ &= -2x \\ &= -2(1) \\ &= -2 \end{aligned}$$



Curve (f_y):
$$\begin{cases} x=t \\ y=y_0 \\ z=f(t, y_0) \end{cases}$$

Curve (f_x):
$$\begin{cases} x=x_0 \\ y=t \\ z=f(x_0, t) \end{cases}$$

Direction vector to tangent line is $a = (x', y', z') = (1, 0, f_x(x_0, y_0))$.

Direction vector to tangent line is $b = (x', y', z') = (0, 1, f_y(x_0, y_0))$.

A normal vector to the tangent plane is $a \times b$.

$$a = (1, 0, f_x(x_0, y_0))$$

$$b = (0, 1, f_y(x_0, y_0))$$

$a \times b =$

$$\begin{vmatrix} 1 & 0 & f_x(x_0, y_0) & 1 & 0 \\ 0 & 1 & f_y(x_0, y_0) & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & f_x(x_0, y_0) & 1 & 0 \\ 0 & 1 & f_y(x_0, y_0) & 0 & 1 \end{vmatrix}$$

$$(-f_x(x_0, y_0), -f_y(x_0, y_0), 1)$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$-f_x(x_0, y_0)(x-x_0) - f_y(x_0, y_0)(y-y_0) + \overset{\uparrow}{f(x_0, y_0)} = 0$$
$$+ 1(\overset{\uparrow}{z} - f(x_0, y_0)) = 0$$

or

$$z = z_0 + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$$

↓

$$f(x_0, y_0)$$

May 20, 2025

Example of Tangent Plane

$$f(x,y) = x^2 - y^2$$

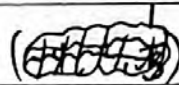
Find the equation of the tangent plane

to the graph of f at $(2, 1, 3)$.

$z = x^2 - y^2$ hyperbolic paraboloid

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f_x(x_0, y_0) = 2x = 4$$



$$f_y(x_0, y_0) = -2y = -2$$

$$z = 3 + 4(x-2) + (-2)(y-1)$$

$$z = 3 + 4x - 8 - 2y + 2$$

$$z = 4x - 2y - 3$$

correct



$$z = 3 + -4(x-2) + 2(y-1)$$

$$z = 3 - 4x + 8 + 2y - 2$$

$$z = -4x + 2y + 3$$

Another example

Find the equation of the tangent plane to the surface

$x^2 - 2y^2 - z^2 = 3$ at the point ~~(1,1,2)~~ (3,1,-2).

↳ 2-sheet elliptic paraboloid

View

$$z = f(x, y) = z(x, y)$$

$$x^2 - 2y^2 - z^2 = 3$$

Equation of the tangent plane is:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= -2 + f_x(3, 1)(x - 3) + f_y(3, 1) \cancel{(y - 1)} (y - 1).$$

Take derivative of both sides of the equation

$$x^2 - 2y^2 - f(x, y)^2 = 3$$

with respect to x :

$$2x - 0 - 2f(x, y)f_x(x, y) = 0$$

implicit

differentiation

Plug $x=3, y=1$.

$$2(3) - 0 - 2f(3, 1)f_x(3, 1) = 0$$

$$6 - 2(-2)f_x(3, 1) = 0$$

$$\frac{4}{4} f_x(3, 1) = \frac{-6}{4}$$

$$f_x(3, 1) = \frac{-3}{2}$$

take derivative

$$x^2 - 2y^2 - f(x,y)^2 = 3$$

with respect to y : $(3, 1, -2)$

$$-4y - 2f(x,y)f_y(x,y) = 0$$

$$-4(1) - 2(-2)f_y(3,1) = 0$$

$$\frac{4}{4} f_y(3,1) = \frac{4}{4}$$

$$f_y(3,1) = 1$$

Eq. of the tangent plane:

$$z = -2 + \left(\frac{-3}{2}\right)(x-3) + (y-1)$$

$$z = -\frac{3x}{2} + \frac{9}{2} + y - 1 - 2$$

$$z = -\frac{3x}{2} + y + \frac{9}{2} - \frac{6}{2}$$

$$z = -\frac{3x}{2} + y + \frac{3}{2}$$

Linear Approximation

$$f(x) = x^2$$



What is the linear function that best

approximates $f(x) = x^2$ near the point $x=1$?

* What we are looking for: $f(x) \approx ax+b$

* tangent line is the best approximate

$f(x) \approx ax+b \rightarrow$ equation of the tangent line.

$$f'(x) = x^2 = 2x$$

$$y = f'(1)(x-1) + f(1)$$

$$f'(1) = 2(1) = 2$$

$$= 2(x-1) + 1$$

$$= 2x - 2 + 1$$

$$= 2x - 1$$

So, $f(x) \approx 2x - 1$ (near $x=1$)

$$f(1.1) = (1.1)^2 = 1.21$$

$$f(1.1) \approx 2(1.1) + 1$$

$$f(1.01) = (1.01)^2 = 1.0201$$

$$\approx 1.2$$

$$f(1.01) \approx 2(1.01) + 1$$

$$\approx 1.02$$

$$f(x,y) = \frac{x}{x+y}$$

What is the linear function that best approximates

$f(x,y)$ near the point $(1,2)$.

$$f(x,y) \approx ax + by + c$$

$$f(x,y) \approx f_x(x,y)(x-1) + f_y(x,y)(y-2) + f(x,y)$$

$$\approx$$

$$f_x(1,2) = \frac{\cancel{x} - 1}{\cancel{x}^2} = \frac{x}{x^2} = \frac{1}{x} = \frac{1}{1} = 1 = 0$$

$$f_y(1,2) =$$

Not sure. I'll finish this one later.

May 21, 2025

Differential of a multivariable function

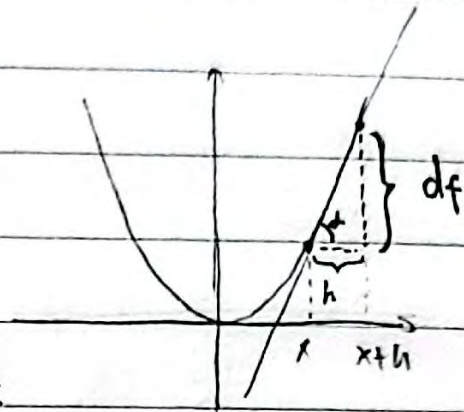
Ex. w/ one variable

$$f(x) = x^2$$

What is the differential of f ?

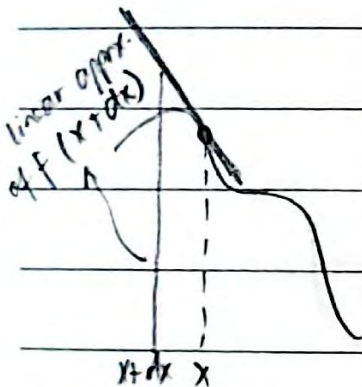
$$df = \underbrace{(\tan \alpha)}_{\text{slope of tangent line}} h = f'(x) h = f'(x) dx$$

slope of
tangent line



* Differential of f is the difference between
the linear approximation of $f(x+dx)$ and $f(x)$.

$$df = f'(x) dx$$



$$f(z) = f(x) + f'(x)(z-x)$$

$$f(x+dx) \approx \underbrace{f(x) + f'(x)dx}_{\text{linear approx.}}$$

nothing to do with x .

Ex: Find an approximate value of $\sqrt{4.02}$

$\sqrt{4.02} \approx \sqrt{4} = 2$. $\leftarrow$ good, but can be better

$$f(x) = \sqrt{x}$$

$$f(4.02) \approx ?$$

$$f(4.02) = f(4 + 0.02)$$

viena as $\downarrow$ $\downarrow$
 x dx

$$f(x+dx) - f(x) \approx (\text{linear approximation of } f(x+dx)) - f(x)$$
$$= df = f'(x) dx = f'(4) 0.02.$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} = 0.25$$

$$df = 0.25(0.02)$$
$$= 0.005$$

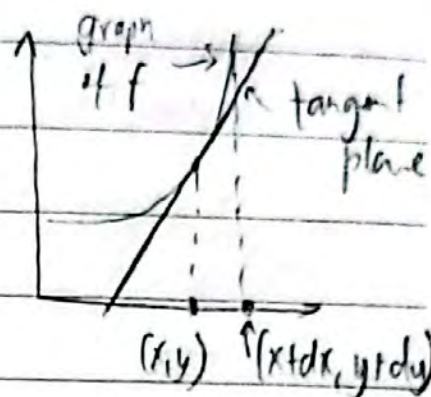
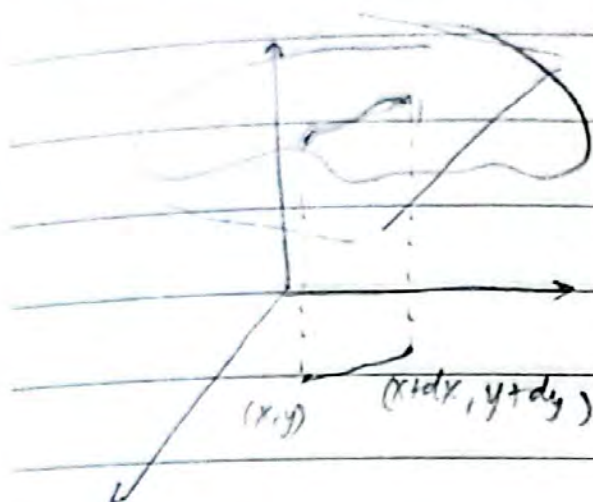
$$\text{so, } f(4.02) - f(4) \approx 0.005.$$

$$\text{So, } f(4.02) \approx \underbrace{f(4)}_2 + 0.005$$
$$= 2.005$$

Multivariable Function

$$f(x, y)$$

$$df = (\text{linear approx. of } f(x+dx, y+dy)) - f(x, y)$$



Linear approximation of $f(x+dx, y+dy)$?

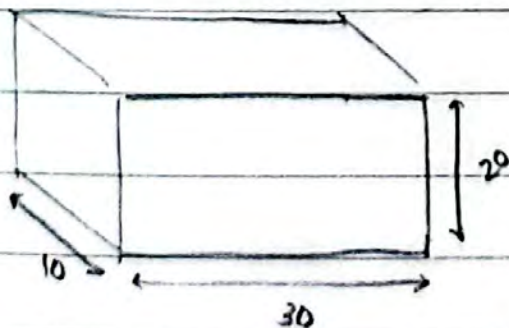
$$f(x+dx, y+dy) \approx f(x, y) + f_x(x, y) dx + f_y(x, y) dy$$

Total Differential

$$df = f_x(x, y) dx + f_y(x, y) dy$$

Partial Differential w.r.t. x & y

Ex:



The possible error when measuring is 1 mm.

What is the maximum possible error of volume?

$$f(x, y, z) = xyz$$

$$f_x = yz$$

$$f_y = xz$$

$$f_z = xy$$

$$= 10 \cdot 20$$

$$= 30 \cdot 20$$

$$= 30 \cdot 10$$

$$df = 200(0.1) + 600(0.1) + 300(0.1)$$

$$= 20 + 60 + 30$$

$$= 110 \text{ cm}^3 //$$

May 22, 2025

Chain Rule

Recall: $f(g(x))$

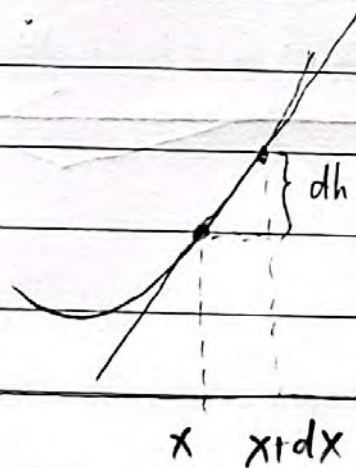
$$\frac{d}{dx} [f(g(x))] = f'(g(x)) g'(x)$$

$$\text{ex. } (e^{x^2})' = e^{x^2}(2x) = 2xe^{x^2}$$

How do we explain this formula?

$$h(x) = f(g(x))$$

$$h' = \frac{dh}{dx}$$



$$dh = h' dx$$

$$\frac{dh}{dx} =$$

$$x \rightarrow x + dx$$

$$h \rightarrow h(x) + dh$$

$$g \rightarrow g(x) + dg$$

$$\begin{aligned}
 h(x+dx) &= f(g(x+dx)) = f(\overbrace{g(x)}^z + \overbrace{dg}^{dz}) \\
 &= f(z) + \underbrace{df}_{f'(z)dz} \\
 &= f(g(x)) + f'(g(x)) dg \\
 &= h(x) + f'(g(x)) g'(x) dx
 \end{aligned}$$

$$\text{So, } dh = f'(g(x)) g'(x) dx$$

$$\text{So, } \frac{dh}{dx} = f'(g(x)) g'(x)$$

Multiple Variables

$$f(x, y), \text{ where } x = x(t), y = y(t)$$

$$h(t) = f(x(t), y(t))$$

If $t \rightarrow t + dt$, then

$$h(t+dt) = h(t) + dh$$

$$x(t+dt) = x(t) + dx$$

$$y(t+dt) = y(t) + dy$$

$$f(x(t+dt), y(t+dt)) = f(x+dx, y+dy)$$

$$= f(x, y) + df$$

$$= f(x, y) + f_x(x, y) dx + f_y(x, y) dy.$$

$$\text{So, } h(t+dt) = h(t) + f_x(x, y) dx + f_y(x, y) dy.$$

$$= h(t) + f_x(x, y) x' dt + f_y(x, y) y' dt.$$

$$= h(t) + \underbrace{(f_x(x,y)x' + f_y(x,y)y')}_{dh} dt$$

So, $\boxed{\frac{dh}{dt} = f_x(x,y)x' + f_y(x,y)y'}$

$$\begin{array}{ll} \sin t & \cos t \\ \cos t & -\sin t \\ -\sin t & -\cos t \\ -\cos t & \sin t \end{array}$$

$$\frac{d}{dt} [f(x(t), y(t))] = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

Ex: Let $f(x,y)$ be a function.

$$\frac{d}{dt} [f(t^2+1, 3t)] = f_x(t^2+1, 3t)2t + f_y(t^2+1, 3t)3$$

$$x(t) = t^2+1$$

$$y(t) = 3t$$

Ex: Find

$$\textcircled{1} \frac{d}{dt} [f(3-t, \ln(t))] = -f_x(3-t, \ln(t)) + f_y(3-t, \ln(t)) \frac{1}{t}$$

$$\textcircled{2} \frac{d}{dt} [g(t^3+t, \sin t, \sqrt{t})] = g_x(t^3+t, \sin t, \sqrt{t})(3t^2+1) +$$

$$g_y(t^3+t, \sin t, \sqrt{t}) \cos t + g_z(t^3+t, \sin t, \sqrt{t}) \frac{1}{2\sqrt{t}}$$

Ex: Suppose $f(x, y)$ satisfies

$$f_x(2, 0) = 1 \quad \text{and} \quad f_y(2, 0) = 3$$

$$\text{Let } h(t) = f(3-t, \ln t).$$

Find $h'(1)$.

$$h'(t) = f_x(3-t, \ln t)(-1) + f_y(3-t, \ln t) \frac{1}{t}$$

$$h'(1) = [f_x(2, 0)](-1) + [f_y(2, 0)] \frac{1}{1}$$

$$= 1(-1) + 3(1)$$

$$= -1 + 3$$

$$= 2$$

Ex: Let $f(x, y)$ be a function. Find

$$f(u^2 - uv, u + v^2)$$

$$\frac{\partial}{\partial u} [f(u^2 - uv, u + v^2)] = f_x(u^2 - uv, u + v^2)(2u - v) + f_y(u^2 - uv, u + v^2)(1)$$

$$\frac{\partial}{\partial v} [f(u^2 - uv, u + v^2)] = f_x(u^2 - uv, u + v^2)(-u) + f_y(u^2 - uv, u + v^2)(2v)$$

$$f(x, y), x = x(u, v), y = y(u, v).$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}$$

Ex: Let $f(x, y, z) = x + yz$

$$x = u + v, y = uv, z = u^2 - v^2.$$

Find $\frac{\partial f}{\partial u}$ in two ways: Chain Rule ①

② Substitute x, y, z into the formula, then differentiate directly.

$$\left[\frac{\partial f}{\partial u} \right]$$

$$\textcircled{1} \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$$

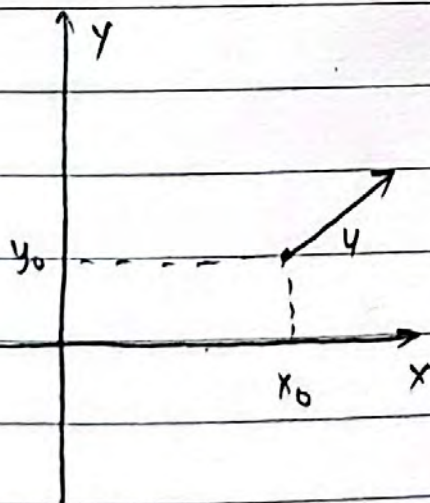
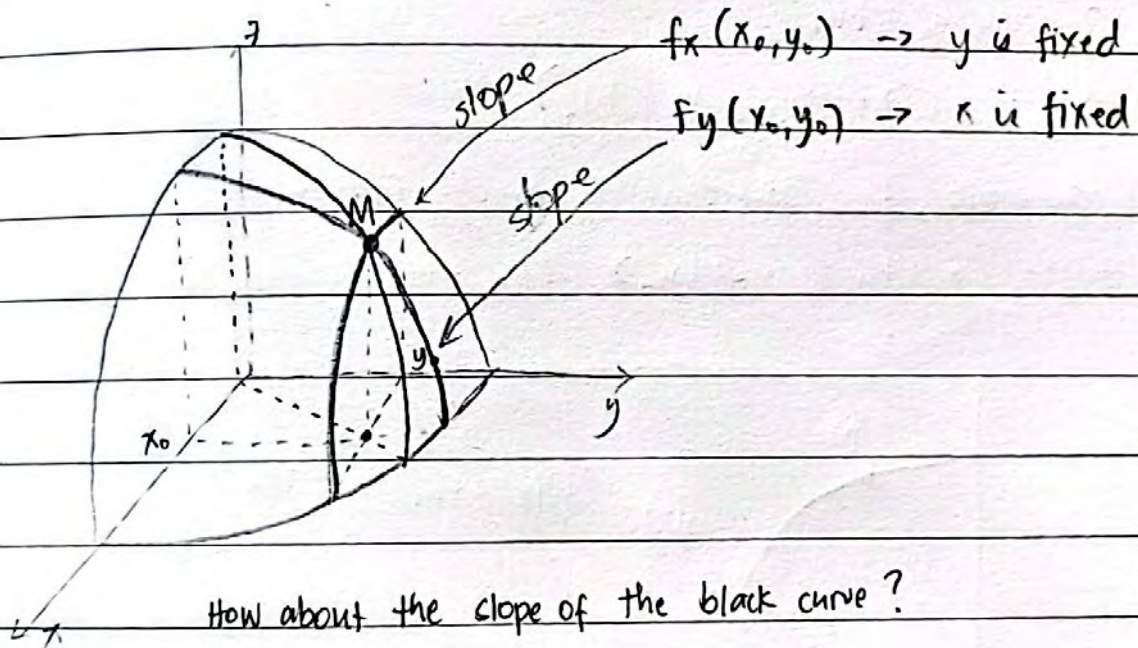
$$= (1)(1) + (z)(v) + (y)(2u)$$

$$= 1 + (u^2 - v^2)v + uv(2u)$$

$$\begin{aligned}
 (2) \quad f(x, y, z) &= x + yz \\
 &= (u+v) + (uv)(u^2 - v^2) \\
 &= u + v + (u^3v - uv^3)
 \end{aligned}$$

May 23, 2025

Directional Derivative



u : unit vector

$$|u| = 1$$

Parametric Equation of the line passing through

(x_0, y_0) with direction vector $u = (u_1, u_2)$ is:

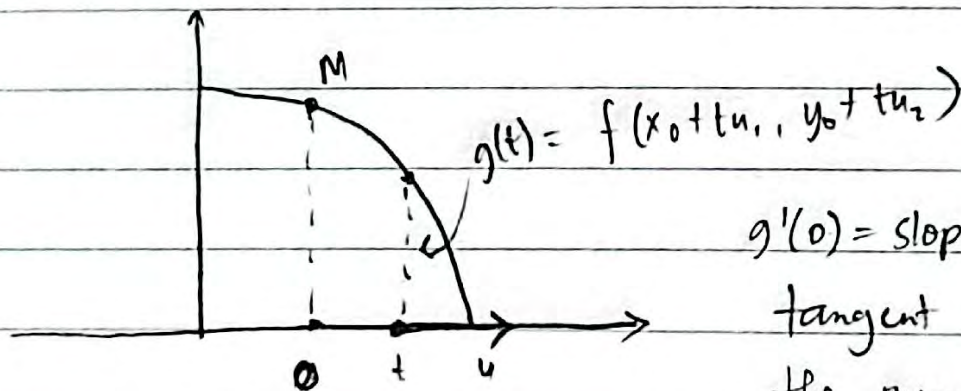
$$\begin{cases} x = x_0 + tu_1 \\ y = y_0 + tu_2 \end{cases} \quad t \in \mathbb{R}$$

Parametric eq. of the black curve is

$$\begin{cases} x = x_0 + tu_1 \\ y = y_0 + tu_2 \\ z = f(x_0 + tu_1, y_0 + tu_2) \end{cases}$$

In other words, the black curve is a vector function

$$r(t) = (x_0 + tu_1, y_0 + tu_2, f(x_0 + tu_1, y_0 + tu_2))$$



$g'(0)$ = slope of the tangent line to the purple curve at M.

$$f_x = 2x + y = 2(1) + 2 = 4$$

$$f_y = x + 4y = (1) + 4(2) = 1 + 8 = 9$$

ERROR

$$\left[\begin{aligned} g'(0) &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \\ &= (1)(u_1) + (1)(u_2) \end{aligned} \right]$$

$$\begin{aligned} g'(0) &= f_x(x_0 + tu_1, y_0 + tu_2)(u_1) + f_y(x_0 + tu_1, y_0 + tu_2)(u_2) \\ &= f_x(x_0, y_0)(u_1) + f_y(x_0, y_0)u_2 \end{aligned}$$

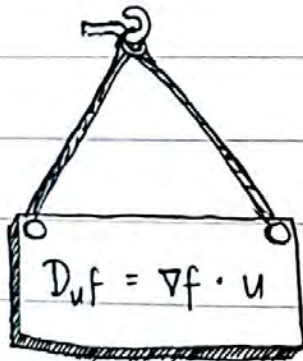
This is called the directional derivative of f at (x_0, y_0) in the direction of vector u .

$$(D_u f)(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

$$= (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot (u_1, u_2)$$

$$= \underbrace{\nabla f(x_0, y_0)}_{\text{gradient vector of } f \text{ at } (x_0, y_0)} \cdot u$$

gradient vector of f at (x_0, y_0)



Ex: Find the directional derivative

of $f(x, y) = x^2 + xy + 2y^2$ at $(1, 2)$

in the direction vector $u = \left(\frac{3}{5}, \frac{4}{5}\right)$

$$D_u f = 4\left(\frac{3}{5}\right) + 9\left(\frac{4}{5}\right) = \frac{12}{5} + \frac{36}{5} = \frac{48}{5}$$

$$f_x = 2x + y$$

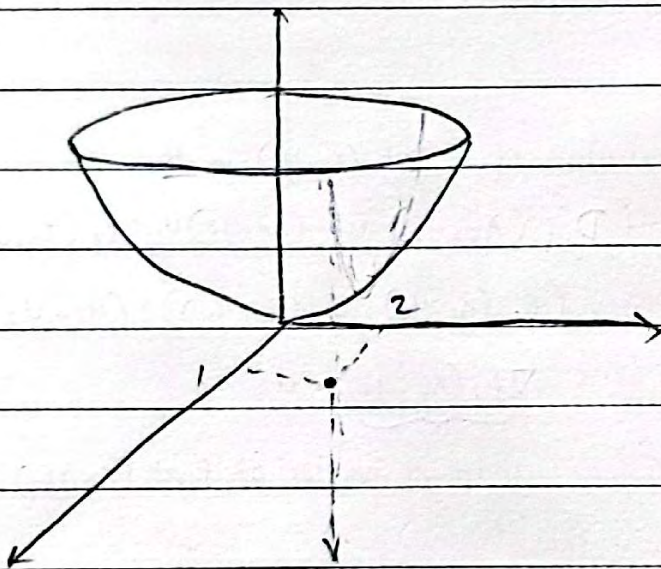
$$f_y = x + 4y$$

$$\nabla f = (f_x, f_y) = (2x + y, x + 4y)$$

$$\nabla f(1, 2) = (2(1) + 2, 1 + 4(2)) = (4, 9)$$

$$D_u f(1, 2) = \nabla f(1, 2) \cdot u$$

$$= (4, 9) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = 4\left(\frac{3}{5}\right) + 9\left(\frac{4}{5}\right) = \frac{48}{5}$$



May 27, 2025

Directional derivative of $f(x, y)$ at (x_0, y_0) in the direction of unit vector u is

$$D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$$

What is direction u that gives the greatest

$D_u f(x_0, y_0)$? In other words, in what

direction (starting at (x_0, y_0)) does

f increase the most?

$$a = \nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

$$D_u f(x_0, y_0) = a \cdot u = |a||u| \cos \theta = |a| \cos \theta$$

fixed $\uparrow$ $\nwarrow$ varies from -1 to 1

$D_u f(x_0, y_0)$ is max

if $\cos \theta = 1$,

is min if $\cos \theta = -1$.

Therefore, $D_u f(x_0, y_0)$ is maximum if $u = \frac{a}{|a|}$,

minimum if $u = -\frac{a}{|a|}$.

The direction that gives the greatest rate of change

$$\text{is } u = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|},$$

and the maximum rate of change is $|\nabla f(x_0, y_0)|$

Ex: Let $f(x, y) = x^2 - xy + y^2$.

(a) Find the gradient vector of f at $(2, -1)$

(b) What are the rates of change of f at $(2, -1)$ in the direction vector $u = (0, 1), (1, 0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{3}{5}, \frac{4}{5})$?

(c) What is the direction that gives the maximum rate of change?

Find the maximum rate of change?

(a) $\nabla f(2, -1) = (5, -4)$

$$f_x = 2x - y$$

$$f_y = -x + 2y$$

(b) $(5, -4) \cdot (0, 1)$

$$0 + (-4) = -4$$

$(5, -4) \cdot (1, 0)$

$$5 + 0 = 5$$

$(5, -4) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\frac{5}{\sqrt{2}} + \left(\frac{-4}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$(5, -4) \cdot (-\frac{3}{5}, \frac{4}{5})$

$$-3 + \frac{-16}{5}$$

$$-\frac{16}{5} + \frac{-15}{5} = \frac{-31}{5}$$

$$(c) \quad u = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|} =$$

Highest rate of change is attained (at) in the direction of

$$u = \frac{(5, -4)}{|(5, -4)|} = \frac{(5, -4)}{\sqrt{41}} = \left(\frac{5}{\sqrt{41}}, \frac{-4}{\sqrt{41}} \right)$$

↓

Highest rate of change is

$$|(5, -4)| = \sqrt{41}.$$

May 28, 2025

$\nabla f(x_0, y_0)$ is perpendicular to the level set
of f that contains (x_0, y_0) .

Optimization Problem

Find min/max of function $f(x, y, \dots)$

Recall: find min/max of $f(x)$ on $[a, b]$.

$$f(x) = |x|(x^2 - x + 1) \text{ on } [-2, 2].$$

Step 1: Find all critical numbers on (a, b) .

place where f is not differentiable

$$\text{or } f'(x) = 0.$$

Step 2: Compare the value of f at the critical numbers
and at a and b .

Find min/max of function $f(x, y, \dots)$

Step 1: Find all critical points $\Rightarrow \nabla f(x_0, y_0) = 0$

place where the tangent plane
is horizontal

Recall: eq. of tangent plane at (x_0, y_0) is

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

For this tangent plane to be horizontal,

we must have :

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

* Revised

Step 1: Find all critical points (x_0, y_0) by

solving the system:

$$\begin{cases} f_x(x_0, y_0) = 0, \\ f_y(x_0, y_0) = 0. \end{cases} \quad (x, y) \in D$$

Step 2: find min/max of f on the boundary of the region D

(denoted by ∂D).

Step 3: Compare the value of f at the critical points

with $\min_{\partial D} f$ and $\max_{\partial D} f$.

Ex: Find min/max of

$$f(x, y) = x^3 + xy^2 - 6x + y^2$$

in the region $D = \{(x, y) : x^2 + y^2 \leq \frac{9}{4}\}$

If $y=0$:

$$3x^2 + 0^2 - 6 = 0$$

$$x = \pm\sqrt{2}$$

If $x=-1$

$$3(-1)^2 + y^2 - 6 = 0$$

$$y = \pm\sqrt{3}$$

5/29/2025

$$f(x, y) = x^3 + xy^2 - 6x + y^2$$

$$= x^3 + x\left(\frac{9}{4} - x^2\right) - 6x + \left(\frac{9}{4} - x^2\right)$$

$$= -x^2 - \frac{15}{4}x + \frac{9}{4}$$

We want to find min/max of

$$\text{function } g(x) = -x^2 - \frac{15}{4}x + \frac{9}{4}$$

$$\text{on } \left[-\frac{3}{2}, \frac{3}{2}\right]$$

$$g'(x) = -2x - \frac{15}{4}$$

$$g'(x) = 0 \text{ when } x = -\frac{15}{8} \notin \left[-\frac{3}{2}, \frac{3}{2}\right]$$

$$\min g = \min \left\{ g\left(-\frac{3}{2}\right), g\left(\frac{3}{2}\right) \right\}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\frac{45}{8} \qquad \qquad -\frac{45}{8}$$

Therefore,

$$\min_{\partial D} f = -\frac{45}{8}, \text{ attained at } (x, y) = \left(\frac{3}{2}, 0\right)$$

$$\max_{\partial D} f = \frac{45}{8}, \text{ attained at } (x, y) = \left(-\frac{3}{2}, 0\right)$$

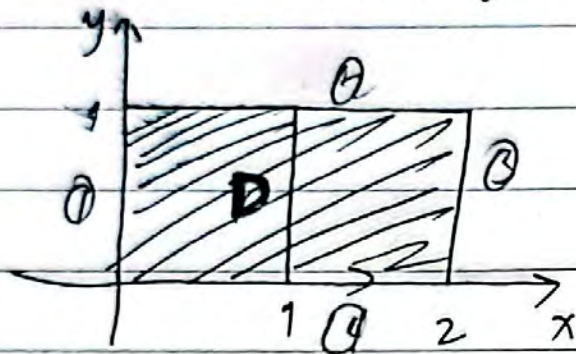
Step 3: Compare $\underbrace{f(\sqrt{2}, 0)}_{-4\sqrt{2}}, \underbrace{f(-\sqrt{2}, 0)}_{4\sqrt{2}}, \underbrace{f(\frac{3}{2}, 0)}_{-\frac{45}{8}}, \underbrace{f(-\frac{3}{2}, 0)}_{\frac{45}{8}}$

$$\max_D f = 4\sqrt{2}, \text{ attained at } (x, y) = (-\sqrt{2}, 0)$$

$$\min_D f = -4\sqrt{2}, \text{ attained at } (x, y) = (\sqrt{2}, 0)$$

Ex: find min/max of $f(x,y) = x^3 + xy^2 - 6x + y^2$

on the ~~rectangle~~ D .
↳ rectangle



critical point $f(\sqrt{2}, 0)$.

$\partial_n(3)$:

$$x=2, 0 \leq y \leq 1$$

$$f(x,y) - f(2,y) =$$

$$= 8 + 2y^2 - 2 + y^2$$

$$= 3y^2 - 4$$

$\partial_n(1)$: $x=0, 0 \leq y \leq 1$

$$f(x,y) = y^2$$

$\partial_n(2)$: $0 \leq x \leq 2, y=1$

$$f(x,y) = 5 + xy - x - 2y$$

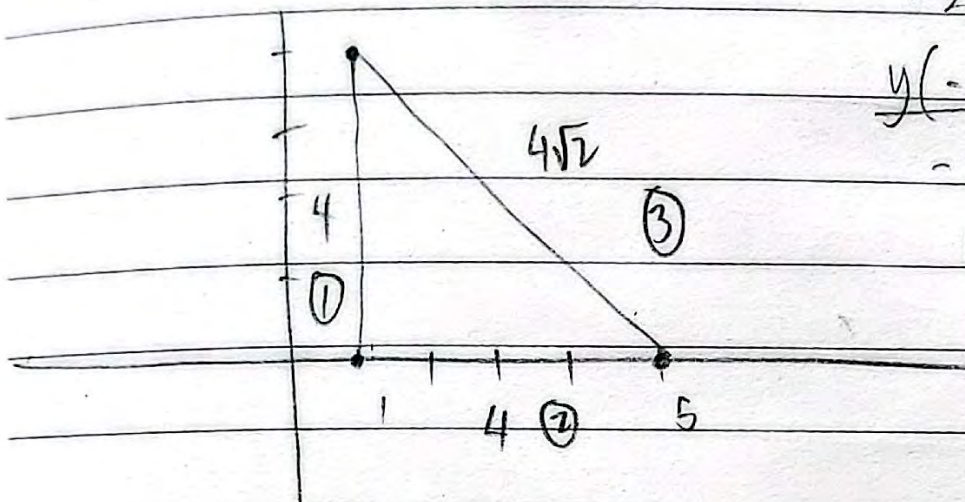
$$D(1,0), (5,0), (1,4)$$

$$5 + xy - x - 2y = 0$$

$$-2y + xy = x - 5$$

$$\frac{y(-2+x)}{-2+x} = \frac{x-5}{-2+x}$$

$$y = \frac{x-5}{-2+x}$$



$$f_x(x,y) = y - 1$$

$$y = 1$$

$$f(2,1) = 3$$

$$f_y(x,y) = x - 2$$

$$x = 2$$

$$m(1): x=1, 0 \leq y \leq 4$$

$$f(x,y) = 5 + y - 1 - 2y$$

$$f(1,0) = 4 \quad \text{max}$$

$$f(x,y) = 4 - y$$

$$f(1,4) = 0 \quad \text{min}$$

$$m(2): 1 \leq x \leq 5, y=0$$

$$f(1,0) = 4 \quad \text{max}$$

$$f(x,y) = 5 - x$$

$$f(5,0) = 0 \quad \text{min}$$

Qn (2)

$$5 + xy - x - 2y$$

$$(1, 4) \quad (5, 0)$$

$$5 - x = 2y - xy$$

$$\text{slope} = -1$$

$$\frac{5-x}{2-x} = \frac{y(2-x)}{2-x}$$

$$y - 0 = -1(x - 5)$$

$$y = -x + 5$$

$$y = 5 - x, 1 \leq x \leq 5$$

$$\begin{aligned} f(x, y) &= 5 + x(5-x) - x - 2(5-x) \\ &= 5 + 5x - x^2 - x - 10 + 2x \\ &= 6x - x^2 - 5 \end{aligned}$$

$$\begin{aligned} f'(x) &= -x^2 + 6x - 5 \\ &= -2x + 6 \end{aligned}$$

$$\begin{aligned} &-9 + 18 - 5 \\ &-1 + 6 - 5 \end{aligned}$$

$$x = 3$$

$$\min_{[1, 5]} g = \min \left\{ \underset{\substack{\vee \\ 4}}{g(3)}, \underset{\substack{\vee \\ 0}}{g(1)}, \underset{\substack{\vee \\ 0}}{g(5)} \right\} = 0 \quad \begin{matrix} (1, 0) \\ (5, 0) \end{matrix}$$

$$\max_{[1, 5]} g = \max \{ g(3) \} = 4 \quad (3, 2)$$

Suppose (x_0, y_0) is a critical point of f .

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ then f attains local min at (x_0, y_0) .

If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ then f attains local max at (x_0, y_0) .

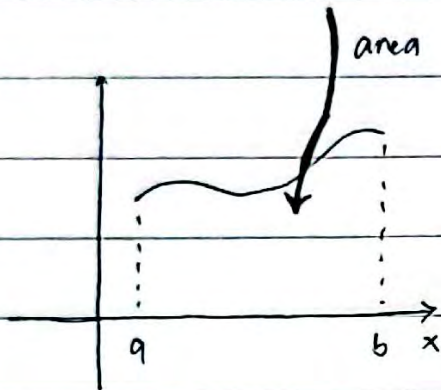
If $D(x_0, y_0) < 0$ then (x_0, y_0) is a saddle point.

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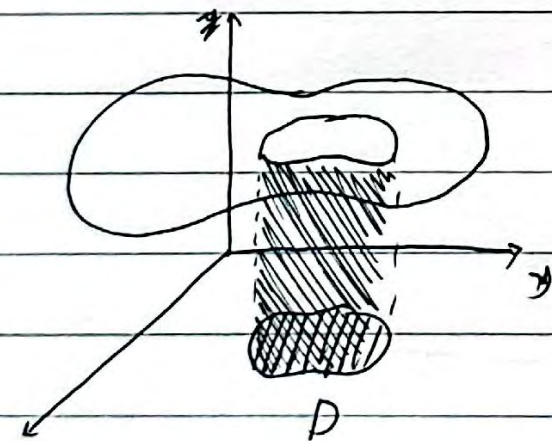
June 1, 2025

Chapter 15 : Integration

Calc I : $\int_a^b f(x) dx$



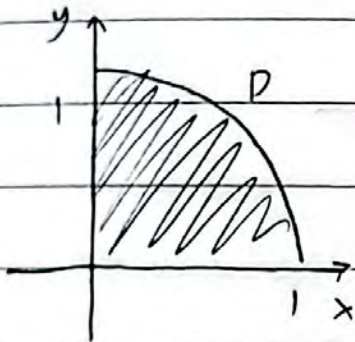
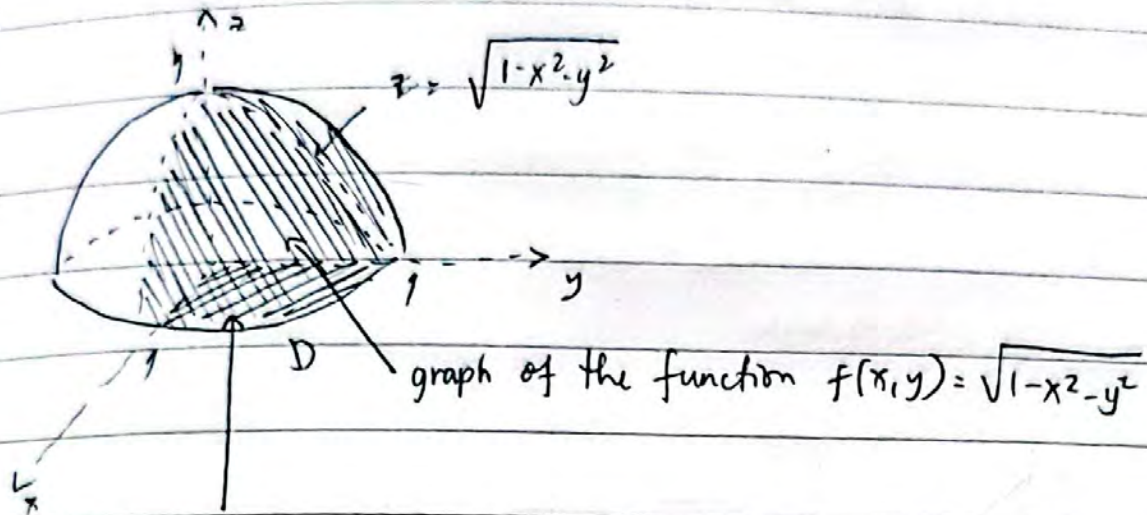
$f(x,y)$



$\iint_D f(x,y) dA$ is the volume of the solid under the graph of f and above the region D in the xy -plane.

double integral $\rightarrow \iint_D f(x,y) dA$ - is called the double integral of f over the region D .

Ex:



$$\iint_D \sqrt{1-x^2-y^2} \, dA = \frac{1}{8} \text{ vol (ball of rad. 1)}$$

$$= \frac{1}{8} \left(\frac{4}{3} \right) \pi = \boxed{\frac{\pi}{6}}$$

Tetrahedron

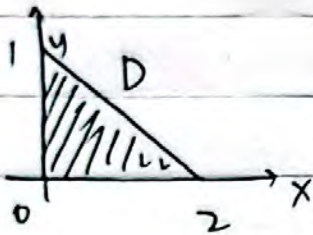


$$\text{Vol} = (\text{base area}) \times (\text{altitude}) \left(\frac{1}{3} \right)$$

Ex: Find the double integral

$$\iint_D (2-x-2y) dA = \frac{1}{3} (2)(1) = \frac{2}{3}$$

where D is the triangle



$$A_{\Delta D} = \frac{1}{2} (2)(1) = 1$$

$$f(x,y) = 2-x-2y$$

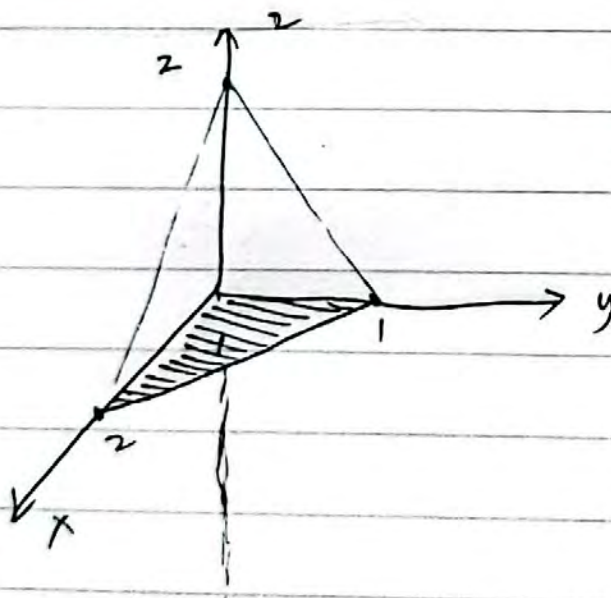
$$z = 2-x-2y$$

$$x^2 + 2y + z = 2$$

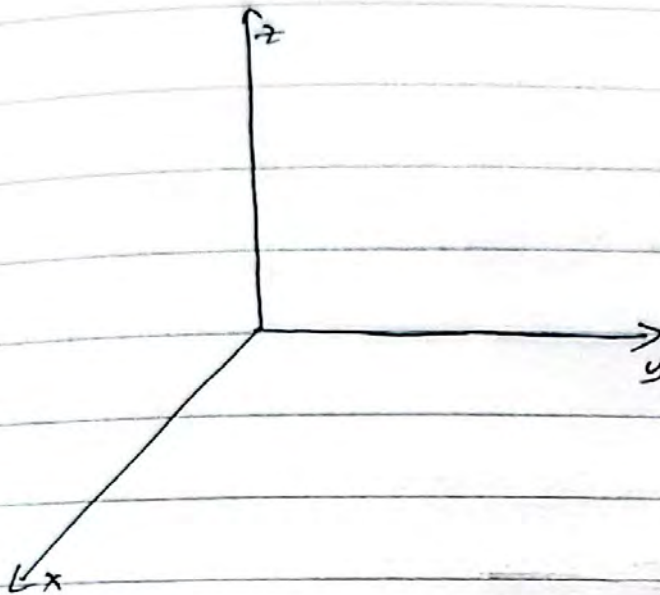
The graph of f is a surface

that has equation

$$z = 2-x-2y$$

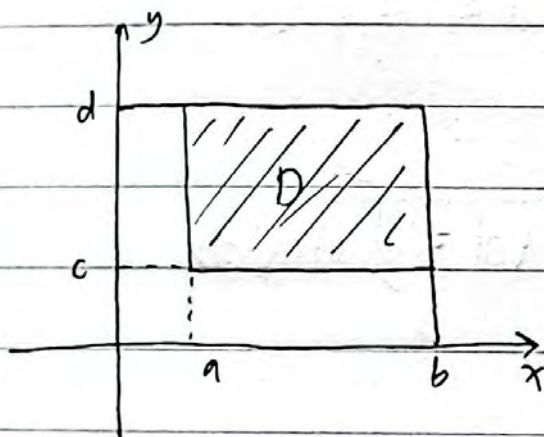


$\iint_D (2-x-2y) dA$ is the volume
of the solid under the
plane $z = 2-x-2y$
and above
the triangle D .



Recall the Cross-section method to find volume :

Analytical method for double integral :



$$D = [a, b] \times [c, d]$$

$$= \{(x, y) : a \leq x \leq b, \\ c \leq y \leq d\}.$$

$$\iint_D f(x, y) dA = ?$$

6/3/2025

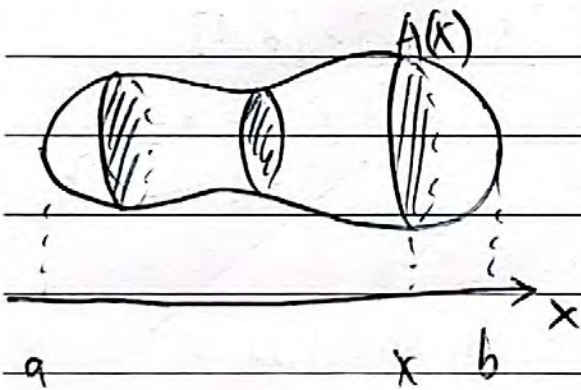
June 2, 2025

* If the solid is above the xy -plane,
it has positive volume integral.

* If the solid is below the xy -plane,
it has negative volume integral.

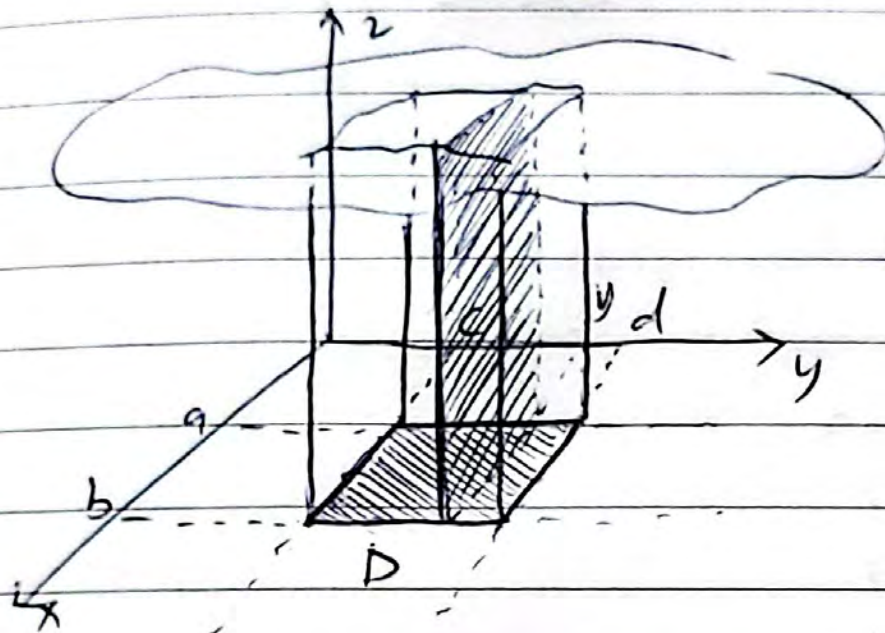
ANALYTICAL METHOD TO FIND VOLUME

Recall the cross-section method to find volume:



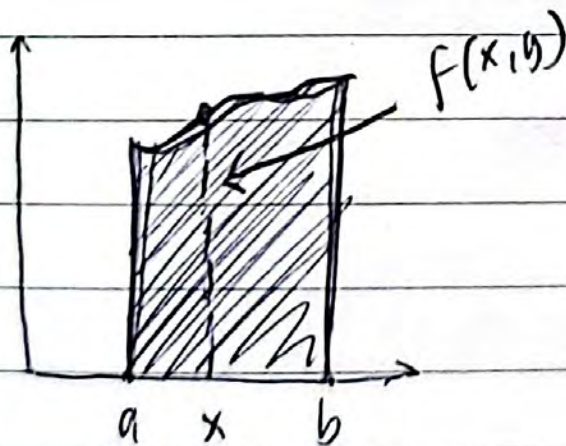
$$\text{Vol} = \int_a^b A(x) dx$$

Consider a simple case where D is a rectangle $D = [a, b] \times [c, d]$



Let $A(y)$ be the area of the cross-section at position y .

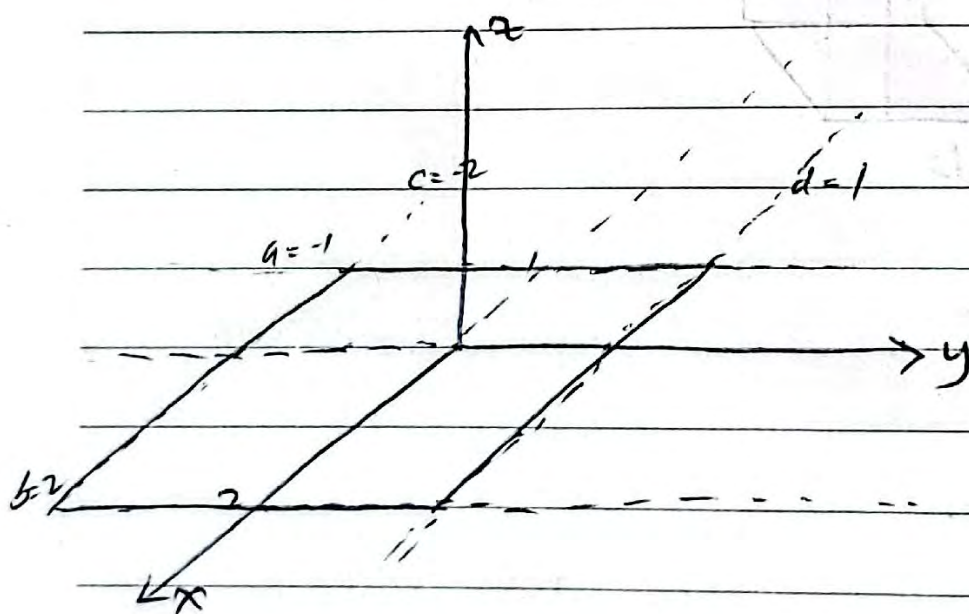
$$\text{vol} = \int_c^d A(y) dy.$$



$$A(y) = \int_a^b f(x, y) dx$$

$$\text{Vol} = \int_c^d \int_a^b f(x,y) dx dy = \iint_{[a,b] \times [c,d]} f(x,y) dA$$

Ex: Find the volume of the solid under the surface $z = x^2 + 2y^2$ and above the rectangle $D = \{(x,y) : -1 \leq x \leq 2, -2 \leq y \leq 1\}$.



$$\begin{aligned} \text{Vol} &= \int_c^d \int_a^b f(x,y) dx dy \\ &= \int_{-2}^1 \int_{-1}^2 (x^2 + 2y^2) dx dy \\ &= \int_{-2}^1 \int_{-1}^2 \frac{x^3}{3} + \frac{2y^2}{(1)} x dy \end{aligned}$$

$$= \left[\left(\frac{(2)^3}{3} + 2y^2(2) \right) - \left(\frac{(-1)^3}{3} + 2y^2(-1) \right) \right]$$

$$= \frac{8}{3} + 4y^2 - \left(\frac{-1}{3} \right) + (-2)y^2$$

$$= \int_{-2}^1 (3 + 6y^2) dy$$

$$= \left(3y + \frac{6y^3}{3} \right) \Big|_{-2}^1$$

$$= \left[\left(3(1) + \frac{6(1)^3}{3} \right) - \left(3(-2) + \frac{6(-2)^3}{3} \right) \right]$$

$$= 3 + \frac{6}{3} - \left((-6) + \frac{6(-8)}{3} \right)$$

$$= 3 + 2 + 6 + 16$$

$$= 5 + 6 + 16$$

$$= 11 + 16$$

$$= \underline{\underline{27}}$$

Recall:

$$\underbrace{\iint_{[a,b] \times [c,d]} f(x,y) dA}_{\text{double integral}} = \underbrace{\int_c^d \int_a^b f(x,y) dx dy}_{\text{iterated integral}} = \int_c^d \int_a^b f(x,y) dy dx$$

$$Vol = \int_{-1}^2 \int_{-2}^1 (x^2 + 2y^2) dy dx$$

$$= \int_{-1}^2 \left(yx^2 + \frac{2y^3}{3} \right) \bigg|_{y=-2}^{y=1} dx$$

$$= \int_{-1}^2 \left[\left(x^2 + \frac{2(1)^3}{3} \right) - \left(x^2 + \frac{2(-2)^3}{3} \right) \right] dx$$

$$= \int_{-1}^2 \left[\left(x^2 + \frac{2}{3} \right) + \left(x^2 + \frac{16}{3} \right) \right] dx$$

$$= \int_{-1}^2 \left[2x^2 + \frac{18}{3} \right] dx$$

$$= \int_{-1}^2 (2x^2 + 6) dx$$

$$= \left. \frac{+3x^3}{3} + 6x \right|_{x=-1}^{x=2}$$

$$= \left. +x^3 + 6x \right|_{x=-1}^{x=2}$$

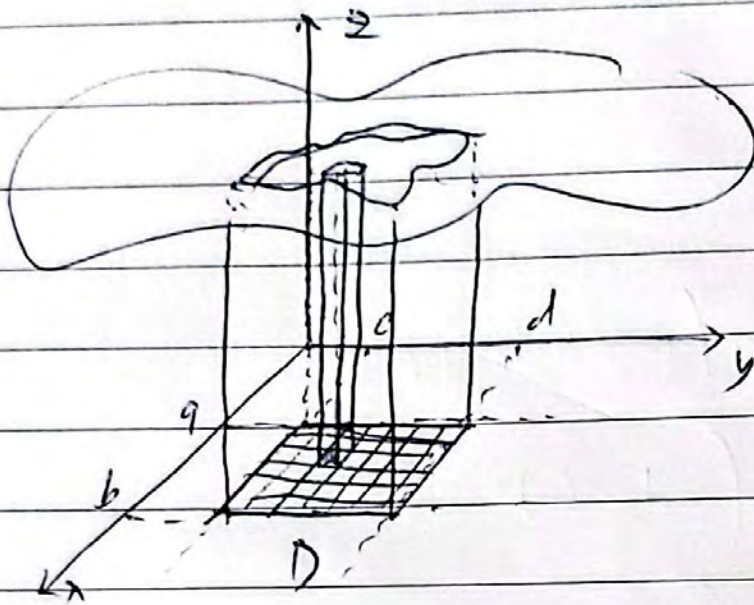
$$= \left((2)^3 + 6(2) \right) - \left((-1)^3 + 6(-1) \right)$$

$$= 8 + 12 - (-1 - 6)$$

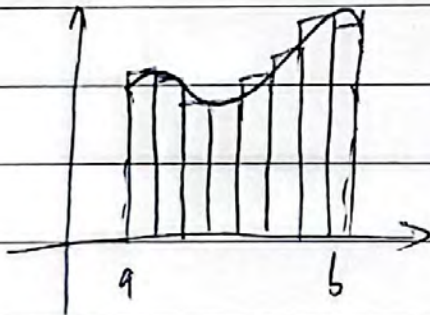
$$= 8 + 12 + (1 + 7)$$

$$= \textcircled{27} //$$

Approximate double integral by Riemann Sum



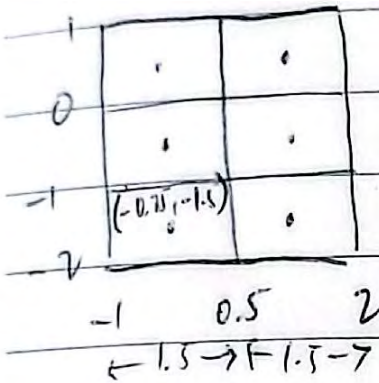
Recall:



$$\int_a^b f(x) dx \approx \text{Sum of areas of the rectangles.}$$

$$\iint_D f(x, y) dA \approx \text{Sum of the volumes of the rectangular prisms.}$$

Ex: $\iint (x^2 + 2y^2) dA$
 $[-1, 2] \times [-2, 1]$



$$f(-0.25, -1.5) \times 1.5 + f(\dots, \dots) \times 1.5 +$$

$$f(\dots, \dots) \times 1.5$$

June 4, 2025

Recall:

$$\iint_{[a,b] \times [c,d]} f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx$$

$$= \int_c^d \int_a^b f(x,y) dx dy$$

Fubini's Theorem

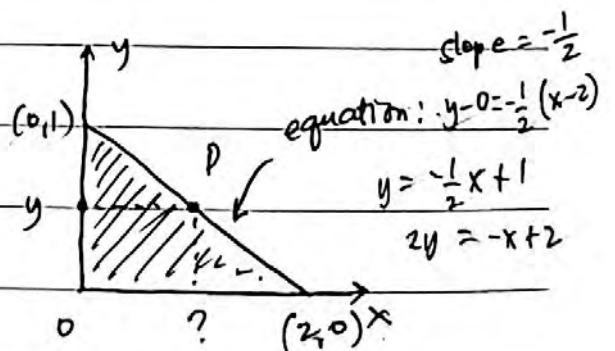
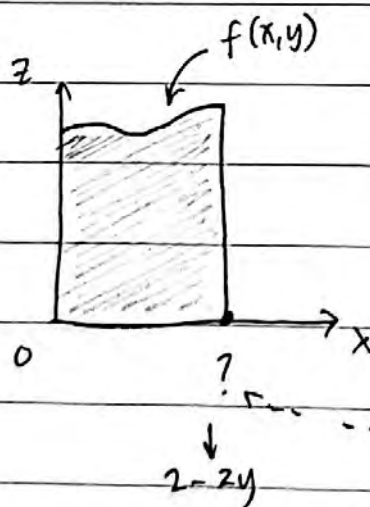
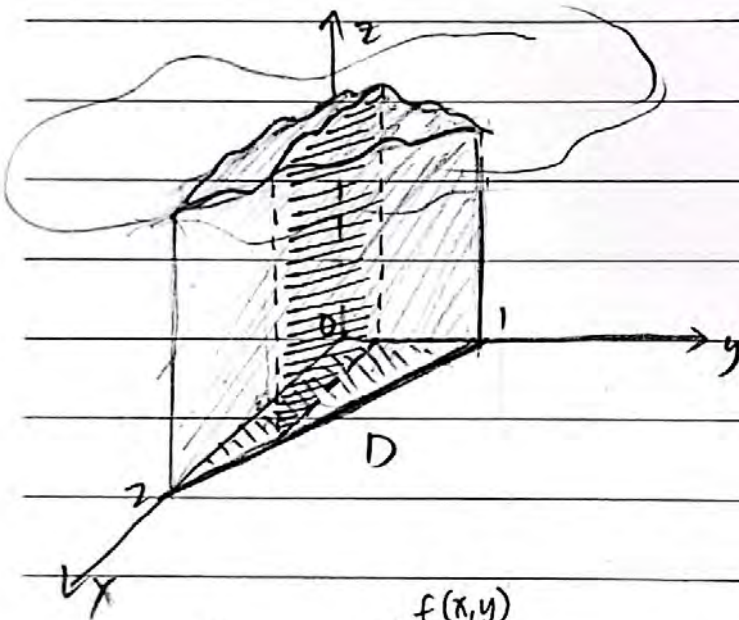
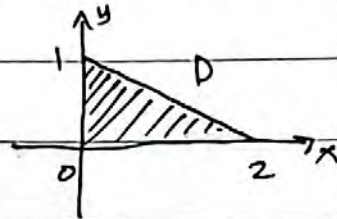
$\sim$ Clairaut's theorem.

If you integrate $f(x,y)$ over a rectangle

$D = [a,b] \times [c,d]$, the order of integration doesn't matter.

What about non-rectangular region D ?

Ex: $\iint_D f(x,y) dA$, where D is a triangle.



$$2-2y$$

$$x = 2 - 2y$$

Area of cross-section:

$$A(y) = \int_0^{2-2y} f(x,y) dx$$

$$\text{Vol} = \int_0^1 A(y) dy$$

$$= \int_0^1 \int_0^{2-2y} f(x,y) dx dy$$

Compute this iterated integral with $f(x,y) = x+y$.

$$= \int_0^1 \int_0^{2-2y} (x+y) dx dy$$

$$= \int_0^1 \left(\frac{x^2}{2} + yx \right) \Big|_0^{2-2y} dy = \int_0^1 \left[\left(\frac{(2-2y)^2}{2} + y(2-2y) \right) - (0) \right] dy$$

$$= \int_0^1 (2 - 4y + 2y^2) + 2y - 2y^2 dy$$

$$= \int_0^1 (2 - 2y) dy$$

$$= \left(2y - \frac{2y^2}{2} \right) \Big|_0^1 = 2 - \frac{2(1)}{2} = 2 - 1 = 1$$

In terms of x :

$$= \int_0^2 \int_0^{-\frac{1}{2}x+1} (x+y) dy dx \quad \left(-\frac{1}{2}x+1\right)\left(-\frac{1}{2}x+1\right)$$

$$= \int_0^2 \left(yx + \frac{y^2}{2} \right) \Big|_{y=0}^{y=-\frac{1}{2}x+1} dx$$

$$= \int_0^2 \left[\left(-\frac{1}{2}x+1\right)x + \frac{\left(-\frac{1}{2}x+1\right)^2}{2} \right] dx$$

$$= \int_0^2 \left[-\frac{1}{2}x^2 + x + \left(\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{2} \right) \frac{1}{2} \right] dx$$

$$= \int_0^2 \left(-\frac{1}{2}x^2 + x + \frac{1}{8}x^2 - \frac{1}{2}x + \frac{1}{2} \right) dx$$

$$= \left(-\frac{1}{2} \left(\frac{x^3}{3} \right) + \frac{x^2}{2} + \frac{1}{8} \left(\frac{x^3}{3} \right) - \frac{1}{2} \left(\frac{x^2}{2} \right) + \frac{1}{2}x \right) \Big|_0^2$$

$$= \left(-\frac{x^3}{6} + \frac{x^2}{2} + \frac{x^3}{24} - \frac{x^2}{4} + \frac{x}{2} \right) \Big|_0^2$$

$$= -\frac{8}{6} + 2 + \frac{8}{24} - 1 + 1 = -\frac{4}{3} + \frac{6}{3} + \frac{1}{3} = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3} \text{ (1)}$$

Summary:

• If $D = \{(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)\}$,

$$\text{then } \iint_D f(x, y) dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx.$$

• If $D = \{(x, y) : c \leq y \leq d, g(y) \leq x \leq h(y)\}$,

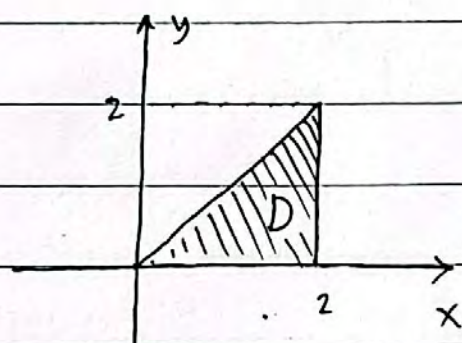
$$\text{then } \iint_D f(x, y) dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y) dx dy.$$

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Ex:

Evaluate $\iint_D e^{x^2} dA$

where D is the triangle



This is wrong!!!

$$\left\{ \begin{aligned} D &= \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 2\} \\ \iint_D e^{x^2} dA &= \int_0^2 \int_0^2 e^{x^2} dx dy \end{aligned} \right\}$$

$$D = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq x\}$$

↑ constrained range

x has a "free" range

$$\iint_D e^{x^2} dA = \int_0^2 \int_0^x e^{x^2} dy dx$$

$$= \int_0^2 (ye^{x^2}) \Big|_{y=0}^{y=x} dx$$

$$= \int_0^2 [xe^{x^2}] dx = \int_0^2 \left(\frac{x}{2x} e^u \right) = \frac{e^u}{2} \Big|_0^4 = \frac{1}{2} (e^4 - 1)$$

$$= \begin{aligned} u &= x^2 \\ du &= 2x dx \end{aligned}$$

x	0	2
u	0	4

$$= \frac{1}{2} (e^4 - e^0)$$

$$= \boxed{\frac{e^4 - 1}{2}}$$

$$D = \{(x, y) : 0 \leq y \leq 2, y \leq x \leq 2\}$$

$$\iint_D e^{x^2} dA = \int_0^2 \int_y^2 e^{x^2} dx dy$$

We are stuck because we can't find an antiderivative of e^{x^2} in an elementary form.

$$u = x^2 \quad \left[\int_0^2 \frac{e^u}{2x} du \right]$$

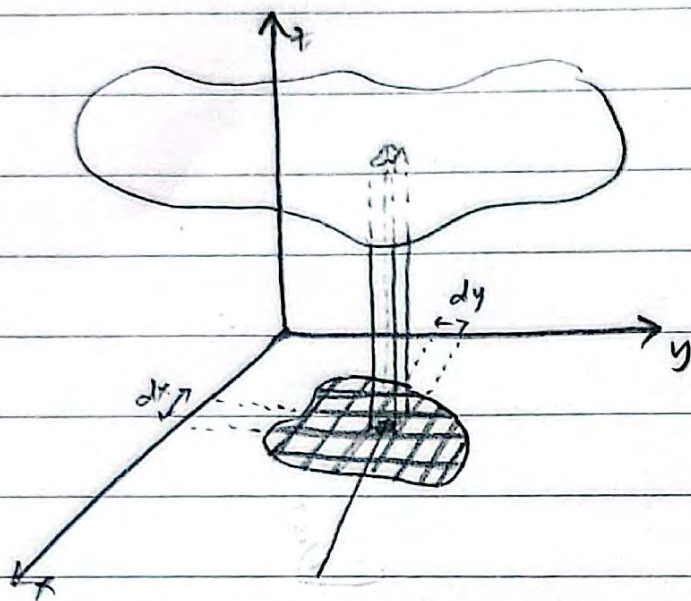
$$du = u' dx$$

$$\frac{du}{2x} = \frac{2x dx}{2x}$$

$$\frac{x/2}{4/2} = \frac{1}{2}$$

Take Note!

$\iint_D 1 dA$ could be tricky if D is a complicated region.



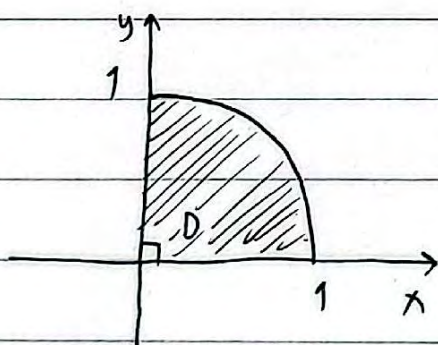
$$dA = dx dy$$

Interpretation:

$$\text{Vol} = \iint_D \underbrace{f(x, y)}_{\text{height}} \pi \underbrace{dA}_{\text{base area}}$$

Consider the case $D = \{(x, y) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$

↑ Polar
Coordinates



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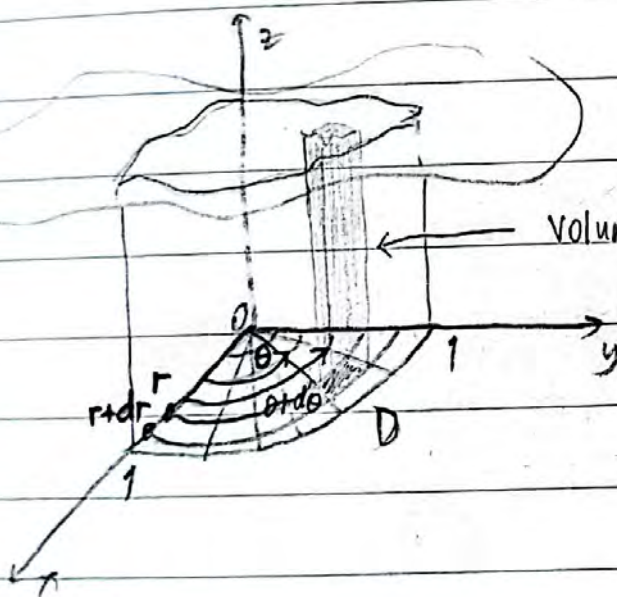
Ex: $\iint_D f(x,y) dA$

$$D = \{(x,y) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$$

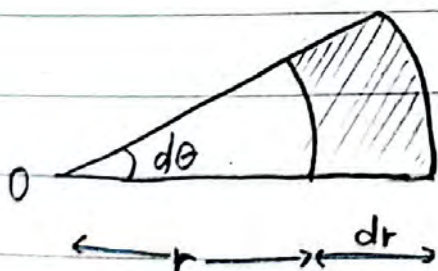


$$D = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$$

$$D = \{(x,y) : 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1-y^2}\}$$



Volume of the "fry" is $f(x,y) \cdot r dr d\theta$
height base area



$$= r dr d\theta + \frac{(dr)^2}{2} d\theta$$

$$\approx r dr d\theta$$

to the shaded area

is approximately ($\approx$) $r dr d\theta$.

Total Volume =

$$\iint_D f(x, y) dA = \iint_{D'} f(x, y) r dr d\theta$$

↳ D' is the region of (r, θ)

$$D' = [0, 1] \times [0, \pi/2]$$

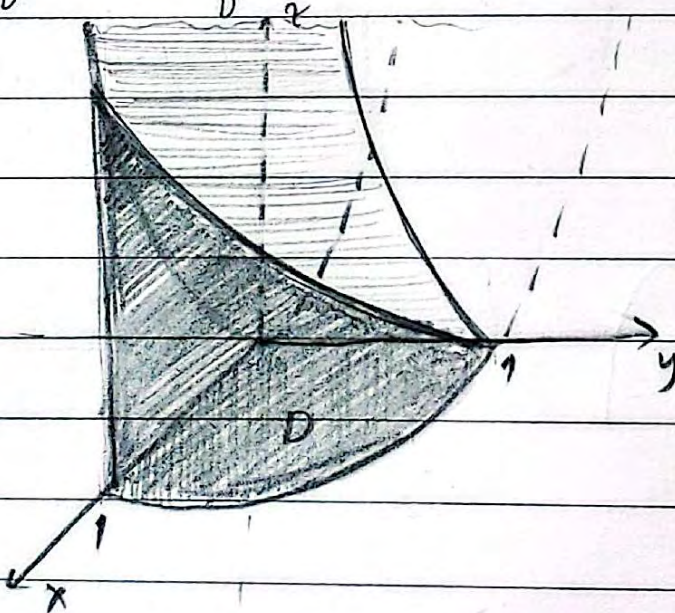
Remember:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r^2 = x^2 + y^2$$

$$\text{Ex: } \iint_D x^2 dA = \iint_{D'} (r \cos \theta)^2 r dr d\theta$$



$$= \iint_{D'} r^3 \cos^2 \theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^3 \cos^2 \theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left. \frac{r^4 \cos^2 \theta}{4} \right|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{(1)^4 \cos^2 \theta}{4} - \frac{(0)^4 \cos^2 \theta}{4} \right] d\theta$$

$$= \int_0^{\pi/2} \frac{\cos^2 \theta}{4} d\theta$$

* Double angle identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$= \int_0^{\pi/2} \frac{1 + \cos 2\theta}{4} d\theta = \int_0^{\pi/2} \frac{1 + \cos 2\theta}{8} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{8} + \frac{\cos(2\theta)}{8} \right] d\theta$$

$$\left(\frac{1}{8}(\theta) + \frac{1}{8} \frac{\sin(2\theta)}{2} \right) \Big|_0^{\pi/2}$$

$$= \frac{1}{8} \left(\theta + \frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/2}$$

$$= \frac{1}{8} \left[\left(\frac{\pi}{2} + \frac{\sin(\pi)}{2} \right) - \left(0 + \frac{\sin(0)}{2} \right) \right]$$

$$= \frac{1}{8} \left[\left(\frac{\pi}{2} + 0 \right) - 0 \right]$$

$$= \frac{1}{8} \left(\frac{\pi}{2} \right) = \left(\frac{\pi}{16} \right)$$

x	0	$\sqrt{3}$
y	1	0

$$y - 1 = \frac{0 - 1}{\sqrt{3} - 0} (x - 0)$$

June 9, 2025

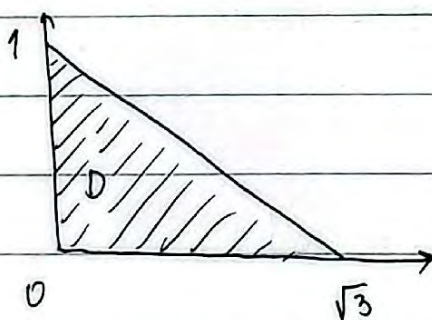
$$(x_1, y_1) = (0, 1)$$

$$(x_2, y_2) = (\sqrt{3}, 0)$$

Ex: Find $\iint_D \frac{1}{\sqrt{x^2+y^2}} dA$

$$y - 1 = \frac{-1}{\sqrt{3}} (x - 0)$$

$$y = -\frac{1}{\sqrt{3}}x + 1$$



$$y = -\frac{1}{\sqrt{3}}x + \frac{\sqrt{3}}{\sqrt{3}}$$

$$\sqrt{3}y = -x + \sqrt{3}$$

$$\sqrt{3}y - \sqrt{3} = -x$$

$$x = \sqrt{3} - \sqrt{3}y$$

$$x = \sqrt{3}(1-y)$$

$$D = \{(x, y) : 0 \leq x \leq \sqrt{3}, 0 \leq y \leq -\frac{1}{\sqrt{3}}x + 1\}$$

$$D = \{(x, y) : 0 \leq y \leq 1, 0 \leq x \leq \sqrt{3}(1-y)\}$$

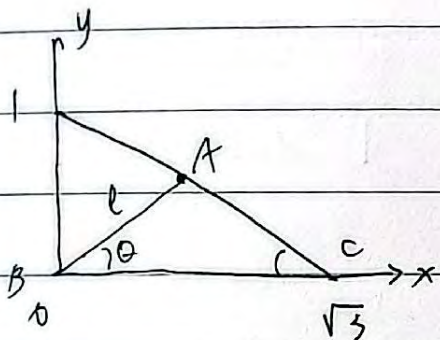
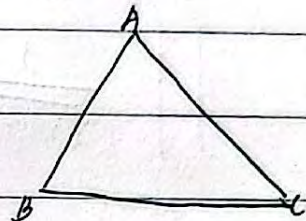
$$= \int_0^{\sqrt{3}} \int_0^{\frac{1}{\sqrt{3}}x+1} \frac{1}{\sqrt{x^2+y^2}} dy dx \quad (1)$$

$$= \int_0^1 \int_0^{\sqrt{3}(1-y)} \frac{1}{\sqrt{x^2+y^2}} dx dy \quad (2)$$

So, Tricky.

We will use Polar Coordinate

Sine law:



$$\tan \hat{C} = \frac{1}{\sqrt{3}}$$

$$\frac{AB}{\sin C} = \frac{BC}{\sin A} = \frac{AC}{\sin B}$$

$$\hat{C} = 30^\circ = \frac{\pi}{6}$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

Sine law:

$$\frac{l}{\sin(\frac{\pi}{6})} = \frac{BC}{\sin \hat{A}} = \frac{\sqrt{3}}{\sin(\pi - \theta - \frac{\pi}{6})}$$

$$0 \leq r \leq l$$

Supplement-angle identity:

$$\sin(\pi - x) = \sin(x)$$

$$= \frac{\sqrt{3}}{\sin(\theta + \frac{\pi}{6})}$$

$$l = \frac{\sin(\frac{\pi}{6}) \sqrt{3}}{\sin(\theta + \frac{\pi}{6})}$$

$$\ell = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sin(\theta + \frac{\pi}{6})}$$

$$D = \left\{ (x, y) : 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq \underbrace{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sin(\theta + \frac{\pi}{6})}}_{\ell} \right\}$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sin(\theta + \frac{\pi}{6})}} \frac{1}{\sqrt{x^2 + y^2}} \cdot r dr d\theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sin(\theta + \frac{\pi}{6})}} \frac{1}{r} \cdot r dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sin(\theta + \frac{\pi}{6})}} 1 dr d\theta$$

$$\begin{aligned} &= \left(\frac{\sqrt{3}}{2(\sin \frac{\pi}{2} + \frac{\pi}{6})} - \frac{\sqrt{3}}{2(\sin \frac{\pi}{6})} \right) \\ &= \left(\frac{\sqrt{3}}{2(\frac{\sqrt{3}}{2})} - \frac{\sqrt{3}}{2(\frac{1}{2})} \right) \\ &= 1 - \sqrt{3} \end{aligned}$$

$$= \int_0^{\pi/2} (r) \left| \frac{\sqrt{3}}{2} \cdot \frac{1}{\sin(\theta + \frac{\pi}{6})} \right| d\theta$$

$$\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{3}}{2} \cdot \frac{1}{\sin(\theta + \frac{\pi}{6})} d\theta$$

$$= \frac{\sqrt{3}}{2} \int_0^{\pi/2} \frac{1}{\sin(\theta + \frac{\pi}{6})} d\theta$$

$$u = \theta + \frac{\pi}{6}$$

$$du = u' d\theta$$

$$du = 1 d\theta$$

$$= \frac{\sqrt{3}}{2} \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \frac{1}{\sin(u)} du$$

$$= \frac{\sqrt{3}}{2} \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \ln \left| \tan\left(\frac{u}{2}\right) \right| du$$

u	$\frac{\pi}{6}$	$\frac{2\pi}{3}$
θ	0	$\frac{\pi}{2}$

$$\frac{\frac{\pi}{6}}{\frac{1}{2}} = \frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2} \left(\ln \left| \tan\left(\frac{2\pi}{3} \cdot \frac{1}{2}\right) \right| - \ln \left| \tan\left(\frac{\pi}{6} \cdot \frac{1}{2}\right) \right| \right)$$

$$= \frac{\sqrt{3}}{2} \left(\ln |+\sqrt{3}| - \ln |\frac{1}{\sqrt{3}}| \right)$$

$$\int \frac{1}{\sin(u)} du = \int \frac{1}{\sin^2(u)} \cdot \sin(u) du$$

u	$\pi/6$	$2\pi/3$
v	$\sqrt{3}/2$	$-1/2$

$$v = \cos u$$

$$dv = v' du = -\sin u du$$

$$= \int \frac{1}{1-v^2} (-dv) = \int \frac{1}{v^2-1} dv$$

$$= \frac{1}{v^2-1} = \frac{1}{(v-1)(v+1)}$$

$$= \frac{A}{v-1} + \frac{B}{v+1}$$

$$v=1 \quad \frac{1}{v^2-1} = \frac{A}{v-1} + \frac{B}{v+1}$$

$$1 = A(v+1) + B(v-1)$$

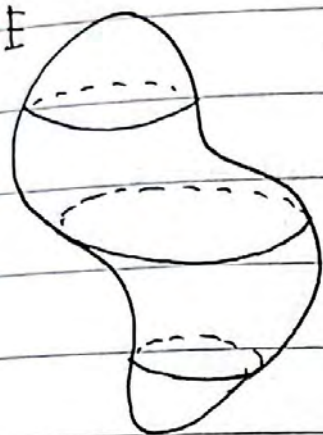
$$v = -1 :$$

$$\frac{1}{-2} = \frac{B(-2)}{-2} \Rightarrow B = -\frac{1}{2}$$

$$v = 1 :$$

$$\frac{1}{2} = \frac{A(2)}{2} \Rightarrow A = \frac{1}{2}$$

Triple Integral

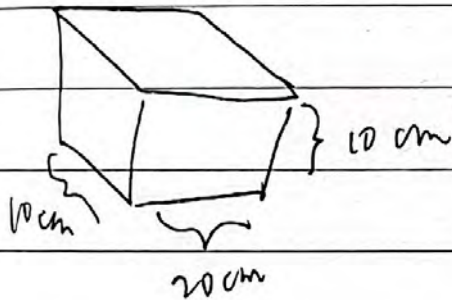


Total mass is

$$\iiint_E f(x, y, z) dV$$

$\int$ triple integral

Mass density $f(x, y, z) \left(\frac{g}{cm^3} \right)$



* homogeneous

Mass distribution is $1g/cm^3$

What is the total mass?

2 kg

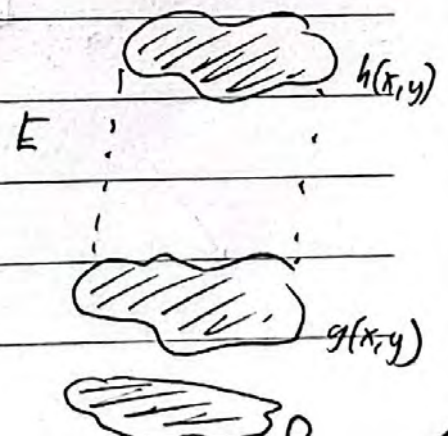
$$\frac{1g}{cm^3} \cdot 2000 cm^3 = 2000g$$

free range

constrained range

$$\text{If: } E = \{(x, y, z) : (x, y) \in D, g(x, y) \leq z \leq h(x, y)\}$$

$$\text{then } \iiint_E f(x, y, z) dV = \iint_{D_{g(x,y)}} \int_{g(x,y)}^{h(x,y)} f(x, y, z) dz dA$$



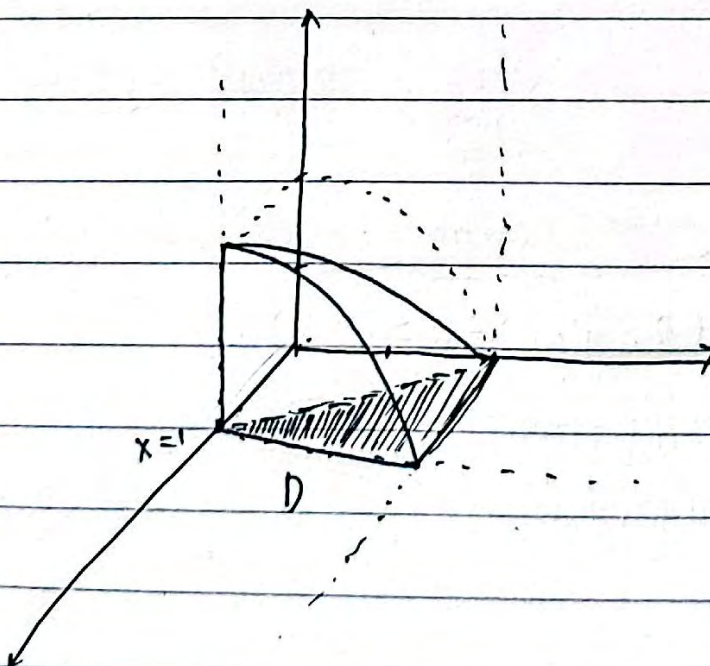
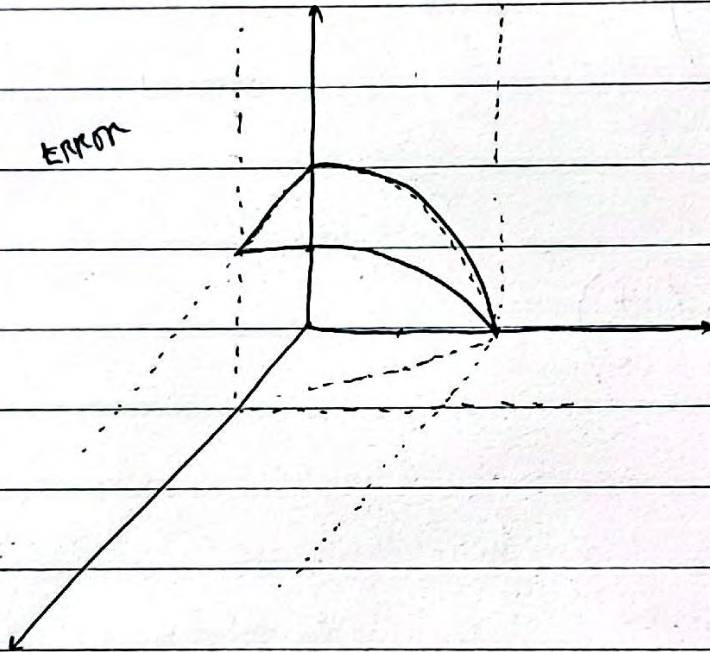
June 10, 2025

Ex: Evaluate $\iiint_E yz \, dV$

E:

where E is the solid bounded by the surfaces

$x=1$, $z=0$, $y=2-2x$, $y^2+z^2=4$ in the first octant.



$$E = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 2-2x, 0 \leq z \leq \sqrt{4-y^2}\}$$

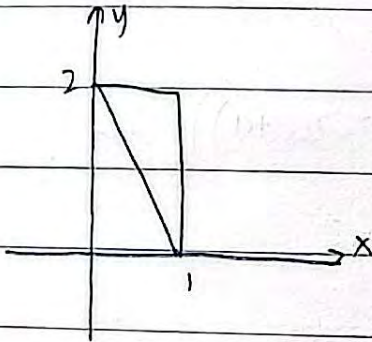
ERROR

$$\int_0^1 \int_0^{2-2x} \int_0^{\sqrt{4-y^2}} yz \, dz \, dy \, dx$$

free range

constrained range

$$E = \{(x, y, z) : (x, y) \in D, 0 \leq z \leq \sqrt{4-y^2}\}$$



$$= \iint_D \int_0^{\sqrt{4-y^2}} yz \, dz \, dA$$

$$= \iint_D y \left(\frac{z^2}{2} \right) \Big|_0^{\sqrt{4-y^2}} dA = \iint_D y \left[\left(\frac{(\sqrt{4-y^2})^2}{2} \right) - \left(\frac{(0)^2}{2} \right) \right] dA$$

$$= \iint_D y \left(\frac{4-y^2}{2} \right) dA = \int_0^1 \int_0^{2-2x} y \left(\frac{4-y^2}{2} \right) dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{2-2x} (4y - y^3) dy \, dx = \frac{1}{2} \int_0^1 \left(\frac{4y^2}{2} - \frac{y^4}{4} \right) \Big|_{y=0}^{y=2-2x} dx$$

$$= \frac{1}{2} \int_0^1 \left(2y^2 - \frac{y^4}{4} \right) \Big|_0^{2-2x} dx$$

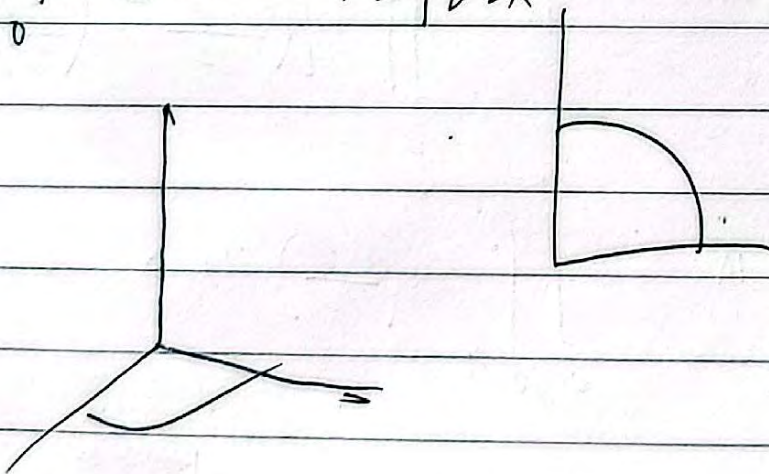
$$= \frac{1}{2} \int_0^1 \left[2(2-2x)^2 - \frac{(2-2x)^4}{4} \right] dx$$

$$= \frac{1}{2} \int_0^1 \left(2(4-8x+4x^2) - (4x^4-16x^3+24x^2-16x+4) \right) dx$$

$$= \frac{1}{2} \int_0^1 \left((8-16x+8x^2) - (4x^4-16x^3+24x^2-16x+4) \right) dx$$

$$= \frac{1}{2} \int_0^1 \int_{2-2x}^2 \frac{y(4-y^2)}{2} dy dx$$

$$= \int_0^1 \left[\frac{1}{2} \left[\frac{2y^2}{2} - \frac{y^4}{4} \right] \right]_{2-2x}^2 dx$$



$$D = \{(x, y) : 0 \leq y \leq 2, \frac{2-y}{2} \leq x \leq 1\}$$

$$= \int_0^2 \int_{\frac{2-y}{2}}^1 \frac{1}{2} (4y - y^3) dx dy$$

$$\frac{2y^3}{2} - \frac{y^3}{1} = \frac{2y^3 - 2y^3}{2} = 0$$

$$= \int_0^2 \frac{1}{2} (4yx - y^3x) \Big|_{\frac{2-y}{2}}^1 dy$$

$$= \int_0^2 \frac{1}{2} \left[(4y(1) - y^3(1)) - (4y(\frac{2-y}{2}) - y^3(\frac{2-y}{2})) \right] dy$$

$$= \int_0^2 \frac{1}{2} \left(4y - y^3 - \left[\frac{8y - 4y^2}{2} - \left(\frac{2y^3 - y^4}{2} \right) \right] \right) dy$$

$$= \int_0^2 \frac{1}{2} \left(2y^2 - \frac{y^4}{2} \right) dy$$

$$\int_0^2 y^2 - \frac{y^4}{4} dy$$

$$4 \times \frac{44}{15} = \frac{176}{15}$$

$$= \frac{y^3}{3} - \frac{1}{4} \left(\frac{y^5}{5} \right) \Big|_0^2$$

$$= \frac{8}{3} - \frac{32}{20} = \frac{20(8) - 3(32)}{60} = \frac{160 - 96}{60} = \frac{64}{60} = \frac{16}{15}$$

$$= \frac{64}{60} = \frac{16}{15}$$

Ex: Evaluate $\iiint_E z \, dv$

where $E = \{(x, y, z) : 0 \leq z \leq 1-y, \sqrt{x} \leq y \leq 1, 0 \leq x \leq 1\}$

$D = \{(x, y) : \sqrt{x} \leq y \leq 1, 0 \leq x \leq 1\}$

$E = \{(x, y, z) : (x, y) \in D, 0 \leq z \leq 1-y\}$

$$\iiint_E z \, dv = \iint_D \int_0^{1-y} z \, dz \, dA$$

$$= \iint_D \left(\frac{z^2}{2} \right) \Big|_{z=0}^{z=1-y} dA$$

$$= \iint_D \frac{(1-y)^2}{2} dA = \iint_D \frac{1}{2} (1-y)^2 dA$$

$$= \int_0^1 \int_{\sqrt{x}}^1 \frac{1}{2} (1-y)^2 dy \, dx$$

$$= \int_0^1 \frac{1}{2} \left(\cancel{\frac{y^3}{3}} (1-2y+y^2) \right) dy \, dx$$

$$= \int_0^1 \frac{1}{2} \left(y - y^2 + \frac{y^3}{3} \right) \Big|_{\sqrt{x}}^1 dx$$

$$= \int_0^1 \frac{1}{2} \left(\left(\frac{1}{3} \right) - \left(\sqrt{x} - x + \frac{x\sqrt{x}}{3} \right) \right) dx$$

June 11, 2025

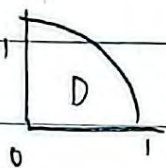
This is the 3D version of polar coordinates.

$$\begin{cases} x = r \cos \theta & (x, y, z): \text{Cartesian Coordinates} \\ y = r \sin \theta & (r, \theta, z): \text{Cylindrical Coordinate} \\ z = z \end{cases}$$

Recall:

$$\iint_D f(x, y) dA = \iint_{D'} f(r \cos \theta, r \sin \theta) r dr d\theta$$

↑ region of (x, y) ↑ region of (r, θ)



$$D = \{(x, y): x^2 + y^2 \leq 1, x, y \geq 0\}$$

$$D' = \{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$$

$$\iiint_E f(x,y,z) dV = \iiint_{E'} f(r \cos(\theta), r \sin(\theta), z) \underbrace{r dr d\theta dz}_{\text{the order of integration}}$$

$\uparrow$ region of (x,y,z)

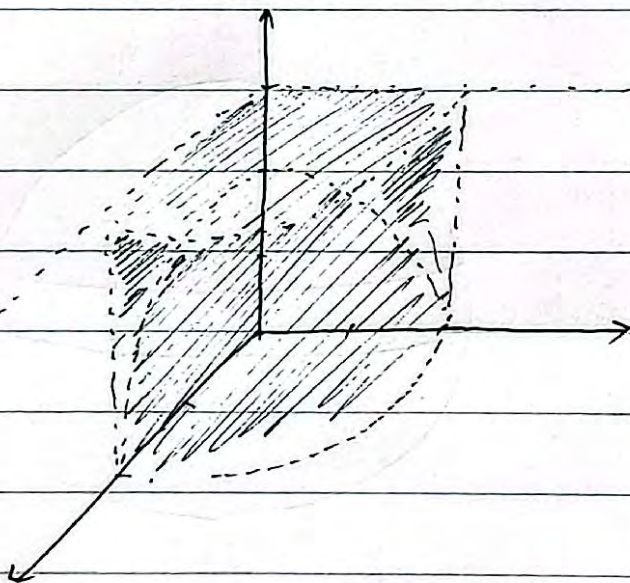
$\uparrow$ region of (r, θ, z)

depends on the range of each variable.

Cylindrical coordinates

Ex: Compute $\iiint_E z dV$ where

E is the solid bounded by the surfaces $x^2 + y^2 + z^2 = 4$,
 $x^2 + y^2 = 4$, $z = 3$ in the first octant.



$$E' = \{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}, \text{ and } \sqrt{4 - (r \cos(\theta))^2 - (r \sin(\theta))^2} \leq z \leq 3\}$$

$$E' = \{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}, \sqrt{4 - r^2} \leq z \leq 3\}$$

$$\int_0^2 \int_0^{\pi/2} \int_{\sqrt{4-r^2}}^3 z \, r \, dz \, d\theta \, dr = \int_0^2 \int_0^{\pi/2} r \left(\frac{z^2}{2} \right) \bigg|_{\sqrt{4-r^2}}^3 d\theta \, dr$$

$$= \int_0^2 \int_0^{\pi/2} r \left[\frac{9}{2} - \frac{4-r^2}{2} \right] d\theta \, dr = \int_0^2 \int_0^{\pi/2} r \left(\frac{r^2+5}{2} \right) d\theta \, dr$$

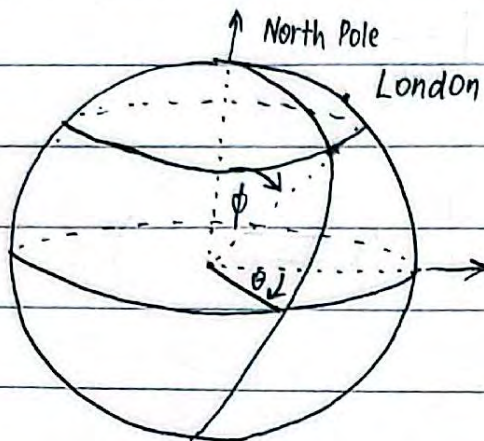
$$= \int_0^2 \int_0^{\pi/2} \frac{1}{2} (r^3 + 5r) d\theta \, dr = \int_0^2 \frac{1}{2} (r^3 \theta + 5r \theta) \bigg|_0^{\pi/2} dr$$

$$= \int_0^2 \frac{1}{2} \left[\left(\frac{\pi r^3}{2} + \frac{5r\pi}{2} \right) - (0) \right] dr$$

$$= \int_0^2 \frac{\pi r^3}{4} + \frac{5r\pi}{2} dr = \frac{1}{4} \left(\frac{\pi r^4}{4} + \frac{5r^2\pi}{2} \right) \bigg|_0^2 = \frac{1}{4} (\pi + 10\pi) = \frac{14\pi}{4}$$

$$= \frac{7\pi}{2}$$

Spherical coordinates



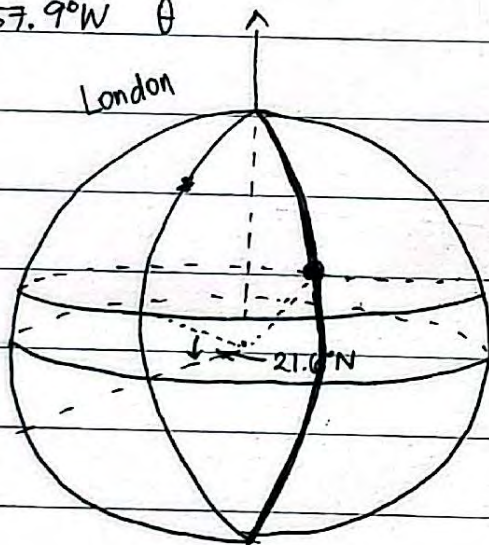
Latitude: $\left[\frac{\pi}{2} - \phi\right]$

Longitude: $\left[\frac{\pi}{2} - \theta\right]$

June 12, 2025

21.6°N $\frac{\pi}{2} - \phi$

157.9°W θ

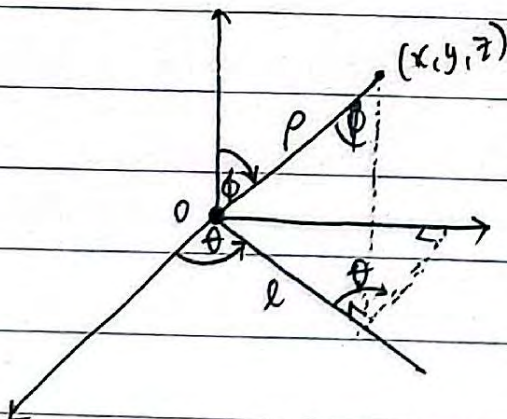


Latitude: $\phi = 90^\circ - 21.6^\circ = 68.4^\circ$

$\theta = -157.9^\circ$ or

202.1°

$\rho = 6370.1 \text{ (km)}$



(ρ, θ, ϕ) : Spherical coordinates

$$z = \rho \cos \phi$$

$$l = \rho \sin \phi$$

$$x = l \cos \theta = \rho \sin \phi \cos \theta$$

$$y = l \sin \theta = \rho \sin \phi \sin \theta$$

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} \geq 0$$

$$0 \leq \phi \leq \pi$$

$$0 \leq \theta \leq 2\pi$$

$$x = 6370.1 (\sin(68.4))(\cos(202.1)) = -5,487.61$$

$$y = 6370.1 (\sin(68.4))(\sin(202.1)) = -2,228.28$$

$$z = 6370.1 (\cos(68.4)) = 2,344.99$$

Ex: Find (ρ, θ, ϕ) when $(x, y, z) = (-1, 2, 3)$

$$\rho = \sqrt{(-1)^2 + (2)^2 + (3)^2} = \sqrt{14}$$

$$\cos \phi = \frac{3}{\sqrt{14}}$$

$$\phi = 36.69^\circ = 0.64 \text{ rad}$$

$$\theta = 116.51^\circ = 2.03 \text{ rad}$$

$$\rho \sin \phi = 1$$

$$\frac{\frac{1}{\sqrt{14}}}{\frac{1}{\sqrt{14}}} = \frac{\sqrt{14}(\sin(36.69))}{\sqrt{14}}$$

$$1 = 2.24$$

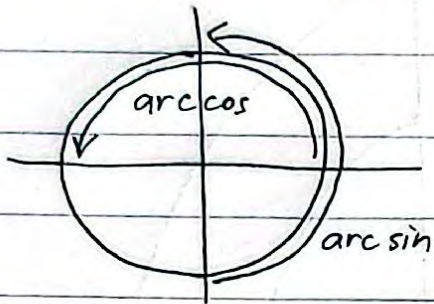
$$(\rho, \theta, \phi) = (\sqrt{14}, 116.51^\circ, 36.69^\circ)$$

$$= (\sqrt{14}, 2.03 \text{ rad}, 0.64 \text{ rad})$$

$$\cos \theta = \frac{x}{\rho} = \frac{-1}{2.24} = 116.51$$

$\arccos$ = gives you an angle on the upper half.

$\arcsin$ = gives you an angle on the right side.



June 13, 2025

Triple Integral with spherical coordinates :

$$\iiint_E f(x, y, z) dV = \iiint f(x, y, z) \boxed{\rho^2 \sin \phi} d\rho d\theta d\phi$$

$$(x, y, z) \rightarrow (\rho, \theta, \phi)$$

$$E \rightarrow E'$$

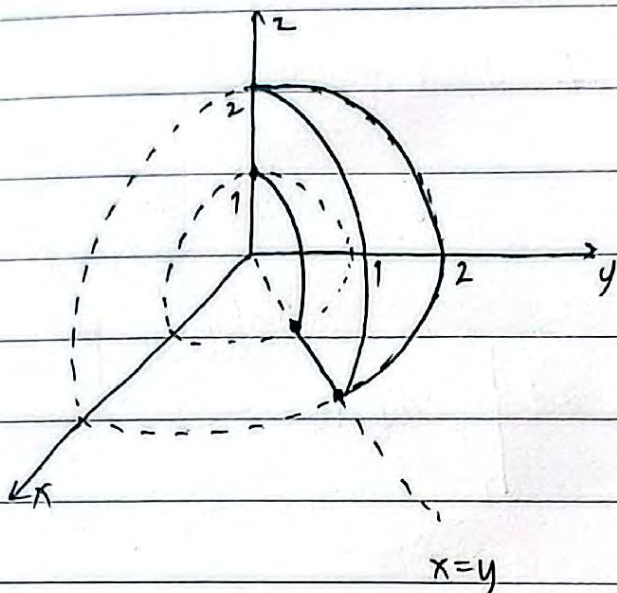
$$\begin{aligned} z &= \rho \cos \phi \\ y &= \rho \sin \phi \sin \theta \\ x &= \rho \sin \phi \cos \theta \end{aligned}$$

Ex: Compute $\iiint_E z dV$

where E is the solid bounded by the surfaces.

$$x=0, y=0, x=y, x^2+y^2+z^2=1, x^2+y^2+z^2=4$$

in the first octant.



Mass density : $f(x, y, z) = z \left(\frac{g}{cm^3} \right)$

$$\iiint_E z \, dV = \text{total mass}$$

$$1 \leq \rho \leq 2$$

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq \phi \leq \frac{\pi}{2}$$

$$E' = \{(\rho, \theta, \phi) : 1 \leq \rho \leq 2,$$

$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2},$$

$$0 \leq \phi \leq \frac{\pi}{2} \}$$

$$\iiint \rho \cos \phi \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

E'



$$= \int_{\pi/2}^{\pi/4} \int_0^{\pi/2} \left. \frac{\rho^4}{4} \cos \phi \sin \phi \right|_1^2 d\phi \, d\theta$$

$$= \frac{1}{4} \int_{\pi/2}^{\pi/4} \int_0^{\pi/2} (16-1) \cos \phi \sin \phi \, d\phi \, d\theta$$

$$= \frac{1}{4} \int_{\pi/2}^{\pi/4} \int_0^{\pi/2} \frac{15}{4} \cos \phi \sin \phi \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \left. \frac{15}{4} \theta \cos \phi \sin \phi \right|_{\pi/4}^{\pi/2} d\phi$$

$$= \int_0^{\pi/2} \left(\frac{15\pi}{16} \cos \phi \sin \phi - \frac{15\pi}{16} \cos \phi \sin \phi \right) d\phi$$

$$= \int_0^{\pi/2} \frac{15\pi}{16} \cos \phi \sin \phi \, d\phi = \int_0^1 \frac{15\pi}{16} u \, du = \left. \frac{15\pi}{16} \left(\frac{u^2}{2} \right) \right|_0^1$$

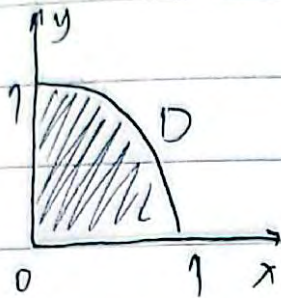
$$u = \sin \phi$$

$$du = \cos \phi \, d\phi$$

ϕ	0	$\pi/2$
u	0	1

$$= \frac{15\pi}{16} \left(\frac{1}{2} \right) = \frac{15\pi}{32}$$

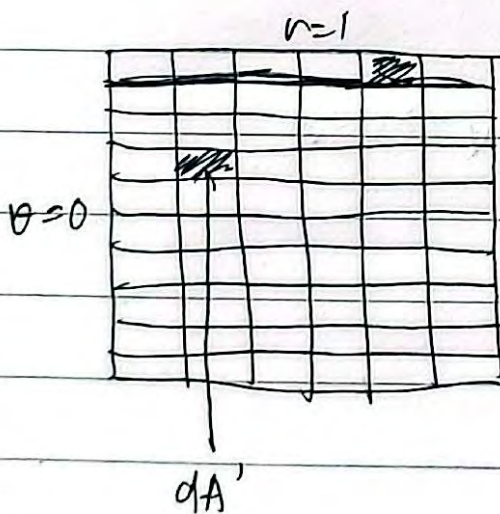
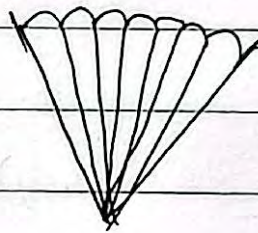
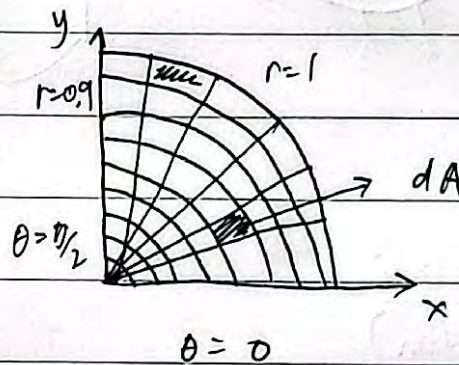
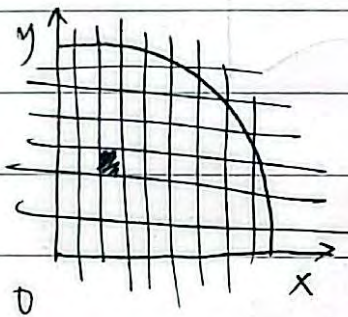
Change of Variables



$$0 \leq r \leq 1$$

$$0 \leq \theta \leq \pi/2$$

$$\iint_D f(x,y) dA$$



$$r=0.9$$

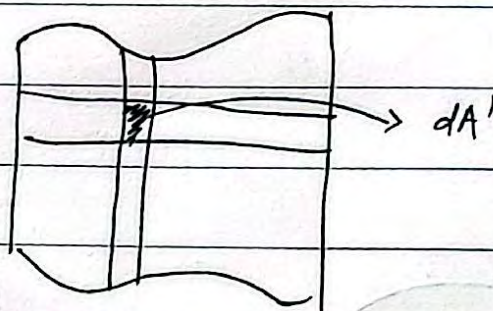
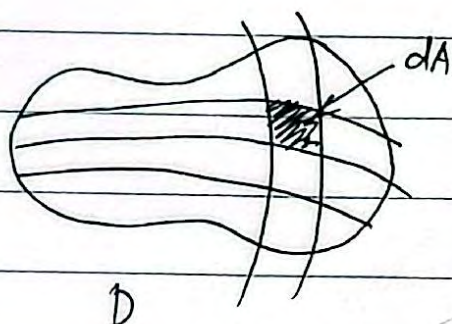
$$\theta = \pi/2$$

$$\frac{dA}{dA'} = r \quad \text{for polar coordinates}$$

$$\iint_D f(x,y) dA = \iint_{D'} f(r \cos \theta, r \sin \theta) r dA'$$

Old variables $(x,y) \in D$

New variables $(u,v) \in D'$



$$\frac{dA}{dA'} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$\frac{\partial(x,y)}{\partial(u,v)}$: Jacobian of the change of variables $(x,y) \rightarrow (u,v)$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = x_u y_v - x_v y_u$$

June 10, 2025

Change of Variables

Double Integral:

$$\iint_D f(x,y) dA = \iint_{D'} f(x(u,v), y(u,v)) \underbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right|}_{dA} du dv$$

$\uparrow$ region of (x,y) $\uparrow$ region of (u,v)

$$\text{Jacobian: } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

Triple Integral:

$$\iiint_E f(x,y,z) dV = \iiint_{E'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \underbrace{\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|}_{\substack{\text{stretching} \\ \text{factor}}} du dv dw$$

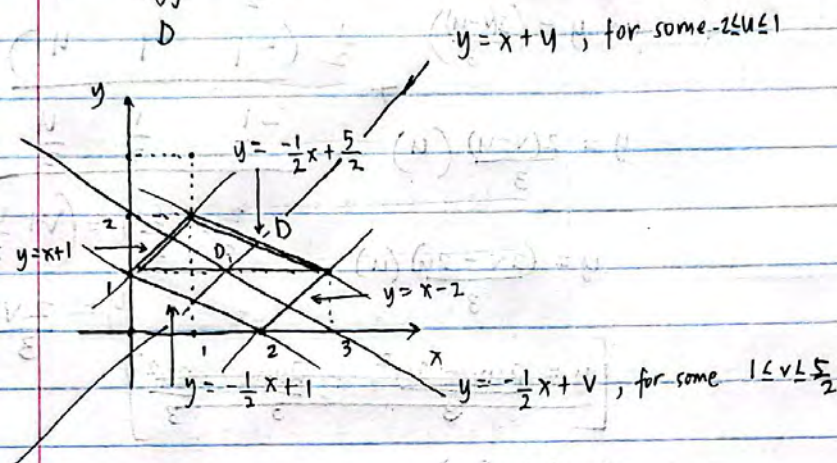
$\uparrow$ region of (x,y,z) $\uparrow$ region of (u,v,w)

$$\text{Jacobian: } \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ w_u & w_v & w_w \end{pmatrix}$$

June 16, 2025

Calculus III

Ex: $\iint_D (2x+y) dA$



$(0,1) \quad (1,2) \quad (2,0) \quad (3,1)$

$y-1 = \frac{2-1}{1-0}(x-0)$

$y-1 = \frac{2-1}{1-0}(x-0)$

$y = x+1$

$(1,2) \quad (3,1) \quad (0,1) \quad (2,0)$

$y-2 = \frac{1-2}{3-1}(x-1)$

$y = -\frac{1}{2}(x-1)+2$

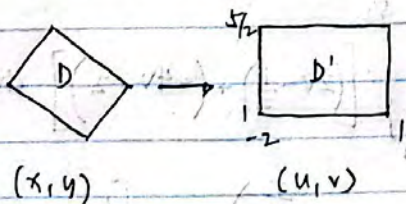
$y = -\frac{1}{2}x + \frac{5}{2}$

Let $u = y-x$ and $v = y + \frac{1}{2}x$

So, $-2 \leq u \leq 1$, $1 \leq v \leq \frac{5}{2}$

Jacobian:

$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$



Solve for x and y in terms of u and v .

$$\begin{aligned} \frac{1}{2}x + y &= v \\ x + y &= u \end{aligned}$$

$$\frac{2}{3} \cdot \frac{3}{2}x = v - u + \frac{2}{3}$$

$$x = \frac{2(v-u)}{3}$$

$$x = \frac{2v}{3} - \frac{2u}{3}$$

$$u = y - \left(\frac{2(v-u)}{3}\right)$$

$$y = \frac{2(v-u)}{3} + \frac{1}{2} \left(-1 \quad 1 \quad u \right)$$

$$y = \frac{(2v-u)}{3} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{u}{2}$$

$$y = \frac{(2v-2u)}{3} + \frac{u}{6}$$

$$y = \frac{2v}{3} + \frac{u}{3}$$

$$\left[y = \frac{2uv - 2u^2}{3} = \frac{2uv}{3} - \frac{2u^2}{3} \right]$$

$$x_u = -2/3$$

$$x_v = 2/3$$

$$y_u = 1/3$$

$$y_v = 2/3$$

$$\text{Jacobian: } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -2/3 & 2/3 \\ 1/3 & 2/3 \end{vmatrix} = -\frac{2}{3} \left(\frac{2}{3} \right) - \left(\frac{2}{3} \right) \left(\frac{1}{3} \right) = -\frac{4}{9} - \frac{2}{9} = -\frac{6}{9} = -\frac{2}{3}$$

$$\int_{-1}^{1/2} \int_{-2}^1 \left[2 \left(\frac{2v}{3} - \frac{2u}{3} \right) + \left(\frac{2v}{3} + \frac{u}{3} \right) \right] \left| -\frac{2}{3} \right| du dv$$

$$\int_{-1}^{1/2} \int_{-2}^1 \left[\left(\frac{4v}{3} - \frac{4u}{3} \right) + \left(\frac{2v}{3} + \frac{u}{3} \right) \right] \left| -\frac{2}{3} \right| du dv$$

$$\int_{-1}^{1/2} \int_{-2}^1 \left[(2v-u) \left(\frac{2}{3} \right) \right] du dv$$

$$\frac{2}{3} \int_{-1}^{1/2} \left(2uv - \frac{u^2}{2} \right) \bigg|_{-2}^1 dv = \int_{-1}^{1/2} \left[\left(2v - \frac{1}{2} \right) + \left(+4v + 2 \right) \right] dv$$

$$= \frac{2}{3} \int_{-1}^{1/2} \left(6v + \frac{3}{2} \right) dv = \frac{2}{3} \left(3v^2 + \frac{3}{2}v \right) \bigg|_{-1}^{1/2} = \frac{2}{3} \left[\left(3 \left(\frac{1}{2} \right)^2 + \frac{3}{2} \left(\frac{1}{2} \right) \right) - \left(3 + \frac{3}{2} \right) \right]$$

$$= \frac{2}{3} \left(\left(3 \left(\frac{25}{4} \right) + \frac{15}{4} \right) - \left(\frac{9}{2} \right) \right)$$

$$= \frac{2}{3} \left(\left(\frac{75}{4} + \frac{15}{4} \right) - \frac{9}{2} \right) = \frac{2}{3} \left(\frac{45}{2} - \frac{9}{2} \right) = \frac{2}{3} \left(\frac{36}{2} \right) = 12 //$$

Chapter 16:

Integral + Curve + vector field

2D Vector Field: $F: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$F(x, y) = (x^2y, 1+2y)$$

3D Vector Field: $F: E \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$F(x, y, z) = (x, y+z, 2y)$$

(x) direction, (y) direction, (z) direction

(v) direction

direction of (v) is

direction of (v) is

direction of (v) is

direction of (v) is

direction of (v) is

direction of (v) is

direction of (v) is

direction of (v) is

June 17, 2025

Math 3/4 Calculus III

Chapter 16:

Integral + vector field

2D Vector Field:

$$F(x,y) = (y, x).$$

3D Vector Field:

$$F(x,y,z) = (x^2, y+z, 1).$$

Visualize $F(x,y) = (y, x)$:
point vector

(x,y)	$F(x,y)$
$(1,1)$	$(1,1)$
$(0,1)$	$(0,1)$
$(1,0)$	$(0,1)$
$(2,0)$	$(0,2)$
$(0,2)$	$(2,0)$
$(1,2)$	$(2,1)$
$(2,1)$	$(1,2)$

Place the vector $(2,1)$ at the point $(1,2)$.

Work:



Force = 200 N

distance = 100 m

work that you have

done is $W = F \cdot s = 200 \times 100 = 20000 \text{ J}$



force = $F = 200 \text{ N}$

distance = 100 m

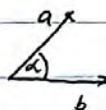
work that your force has done is

$W = F \cdot s \cdot \cos(\alpha)$



Work done by the force field (vector field) is

$F \cdot s \cdot \cos(\alpha) = \vec{F} \cdot \vec{s}$



$a \cdot b = |a||b| \cos(\alpha)$

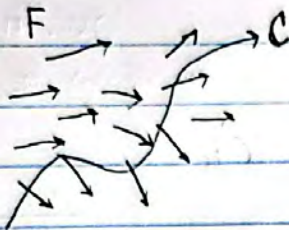
Constant Vector Field:

$F(x, y) = (1, 1)$



* In a constant vector field $\vec{F}$,
the work done by $\vec{F}$ along a
line segment from A to B is

$W = \vec{F} \cdot \vec{AB}$

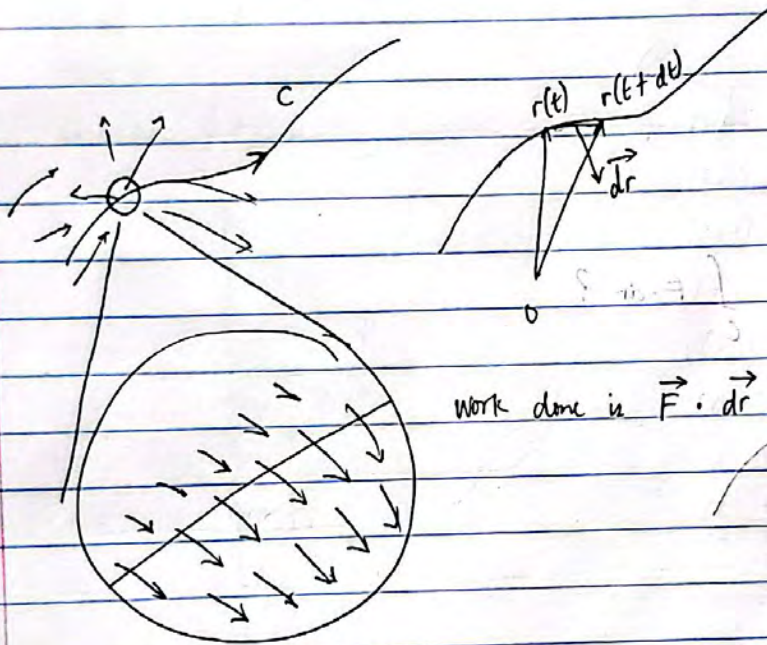


vector field is not constant,
trajectory is not straight.

Work =

C is parametrized by

$$r(t) = (x(t), y(t)), \quad a \leq t \leq b.$$



work done is $\vec{F} \cdot d\vec{r}$

Total work done (w) by $\vec{F}$ along the curve (C) is

$$\int_C \vec{F} \cdot d\vec{r}$$

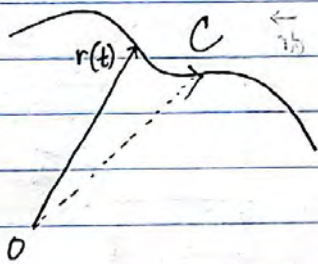
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The work done by vector field $\vec{F}$ along the curve C is

$$\int_C \vec{F} \cdot d\vec{r}$$



How do we compute $\int_C \vec{F} \cdot d\vec{r}$?



C is parametrized by $\vec{r}(t) = (x(t), y(t))$

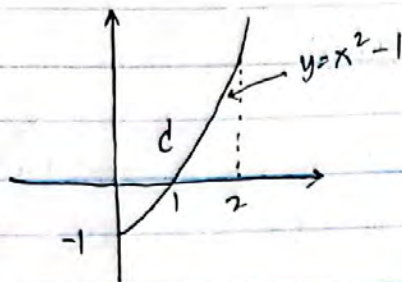
$(a \leq t \leq b)$

$$\vec{r} = \vec{r}(t)$$

$$d\vec{r} = \vec{r}'(t) dt$$

$$\text{So, } \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{r}'(t) dt$$

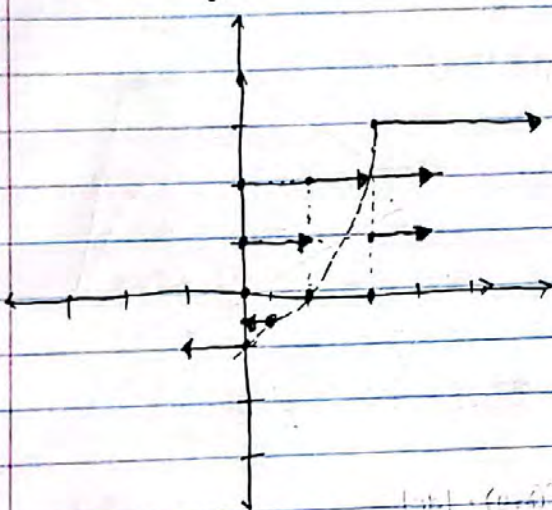
Ex:



$$F(x, y) = (y, 0)$$

Draw F

compute $\int_C F \cdot dr$



$$(x, y) \quad (y, 0)$$

$$(1, 0) \quad (0, 0)$$

$$(0, 1) \quad (1, 0)$$

$$(2, 0) \quad (0, 0)$$

$$(0, 2) \quad (2, 0)$$

$$(1, 2) \quad (2, 0)$$

$$(2, 1) \quad (1, 0)$$

$$(0, -1) \quad (-1, 0)$$

$$(2, 3) \quad (3, 0)$$

$$\left(\frac{1}{2}, -\frac{1}{2}\right) \quad \left(-\frac{1}{2}, 0\right)$$

$$\left(\frac{1}{2}, -\frac{1}{2}\right) \quad \left(-\frac{1}{2}, 0\right)$$

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$$\left(\frac{1}{2}, -\frac{1}{2}\right) \quad \left(-\frac{1}{2}, 0\right)$$

$$\left(\frac{1}{2}, -\frac{1}{2}\right) \quad \left(-\frac{1}{2}, 0\right)$$

$$C: y = x^2 - 1$$

$$\text{parametrization of } C: \begin{cases} x = t \\ y = t^2 - 1 \end{cases}, 0 \leq t \leq 2$$

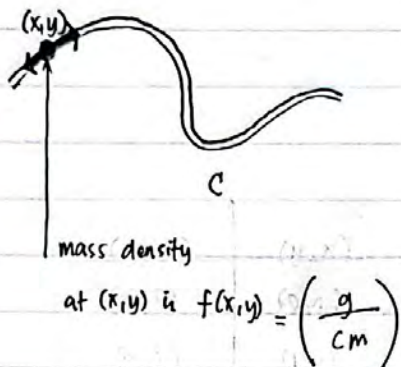
$$\frac{t^3}{3} - t \Big|_0^2 = \frac{2^3}{3} - 2 = \frac{8}{3} - 2 = \frac{2}{3}$$

$$(y)(1) + (0)(2t) = y$$

$$r(t) = (x(t), y(t)) = (t, t^2 - 1)$$

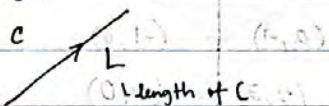
$$\begin{aligned} \int_C F \cdot dr &= \int_0^2 F \cdot r' dt = \int_0^2 (y, 0) \cdot (1, 2t) dt = \int_0^2 y dy = \int_0^2 (t^2 - 1) dt \\ &= \left[\frac{t^3}{3} - t \right]_0^2 = \frac{8}{3} - 2 = \frac{2}{3} \end{aligned}$$

Integral of a scalar function along a curve!



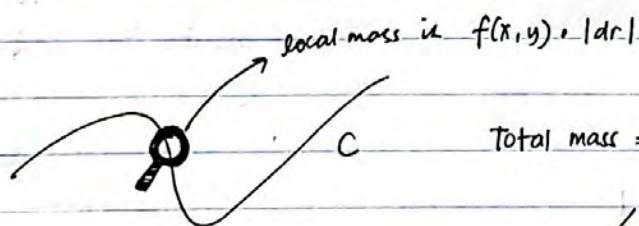
What is the total mass?

Imagine



$f(x, y)$ is constant K

Total mass = LK



$$\text{Total mass} = \int_C f(x, y) |dr|$$

$\downarrow$
 $r'(t) dt$

$$|dr| = |r'(t) dt| = |r'(t)| dt$$

$$= \int_a^b f(x, y) \underbrace{|r'(t)|}_{\text{length}} dt$$

June 20, 2025

Line

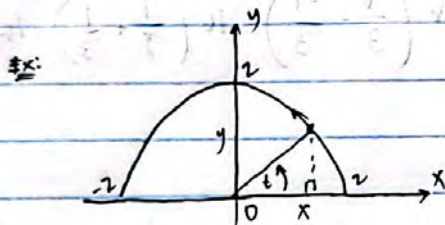
* Integral of vector field along a curve:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}' dt$$

Line

* Integral of scalar function along a curve:

$$\int_C f(|d\mathbf{r}|) = \int_a^b f(|\mathbf{r}'|) dt$$



Mass density is $f(x, y) = x^2 y$ (g/cm)

Find the total mass... of this wire.

Parametrization of C:

$$\mathbf{r}(t) = (x(t), y(t)) = (2 \cos(t), 2 \sin(t)) \quad 0 \leq t \leq \pi$$

$$\mathbf{r}'(t) = (x'(t), y'(t)) = (-2 \sin(t), 2 \cos(t))$$

$$\int_0^\pi (2 \cos(t))^2 (2 \sin(t)) \cdot |(-2 \sin(t), 2 \cos(t))| dt$$

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{(-2 \sin(t))^2 + (2 \cos(t))^2} \\ &= \sqrt{4 \sin^2(t) + 4 \cos^2(t)} = \sqrt{4 \cdot (\sin^2(t) + \cos^2(t))} = \sqrt{4 \cdot 1} = 2 \\ &= 2 \end{aligned}$$

$$= 2 \int_0^\pi (2 \cos(t))^2 (2 \sin(t)) dt = 2 \int_0^\pi 4 \cos^2(t) (2 \sin(t)) dt = 1 \int_0^\pi 8 \cos^2(t) \sin(t) dt$$

$$= 16 \int_0^{\pi} \cos^2(t) \sin(t) dt$$

$$u = \cos(t)$$

$$du = u' dt$$

$$du = -\sin(t) dt$$

t	0	π
u	1	-1

$$= dt = \frac{du}{-\sin(t)}$$

$$= 16 \int_1^{-1} u^2 \frac{du}{-1} = 16 \int_{-1}^1 u^2 du = 16 \left[\frac{u^3}{3} \right]_{-1}^1$$

$$= 16 \left(\frac{1}{3} - \frac{-1}{3} \right) = 16 \left(\frac{1}{3} + \frac{1}{3} \right) = 16 \left(\frac{2}{3} \right) = \frac{32}{3}$$

Conservative Vector Fields

Vector Field F is conservative if there exists a scalar function ϕ such that $F = \nabla \phi$.

Example:

1) $F(x, y) = (x, y)$ is conservative because

$$F = \nabla \phi, \text{ where } \phi(x, y) = \frac{x^2 + y^2}{2}$$

$$\phi_x = x$$

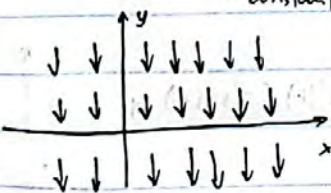
$$\phi_y = y$$

$$\nabla \phi = (x, y) = F$$

ϕ is called a potential function of vector field F .

2) $F(x, y) = (0, -g)$ ← gravitational vector field.

↖ constant.



F is conservative with potential function

$$\phi(x, y) = -gy$$

June 23, 2025

Math 314

Ex:

$$f(x,y) = 3x^2 + y$$

$$F(x,y) = (6x, 1) = \nabla \phi$$

$$\frac{e^x}{\sin y} = \phi(x,y)$$

$$\nabla \phi = \left(\frac{e^x}{\sin y}, \right)$$

Ex: Is $F(x,y) = (y, -x)$ conservative?

Textbook:

$$F(x,y) = y\vec{i} + (-x)\vec{j}$$

$$\frac{y^2}{2} - \frac{x^2}{2} ?$$

Suppose F is conservative: $F = \nabla \phi$.

$$(y, -x) = (\phi_x, \phi_y)$$

$$\text{So, } \phi_x = y, \phi_y = -x$$

$$\text{So, } \phi_{xy} = 1, \phi_{yx} = -1.$$

Since, $\phi_{xy} \neq \phi_{yx}$, it is not conservative.

Therefore, F is not conservative.

Ex: Is $F(x, y) = (6xy^2, 6x^2y + 1)$ conservative?

Suppose F is conservative: $F = \nabla \phi$

$$(6xy^2, 6x^2y + 1) = (\phi_x, \phi_y)$$

$$\text{so, } \phi_x = 6xy^2, \quad \phi_y = 6x^2y + 1.$$

$$\text{So, } \phi_{xy} = 12xy, \quad \phi_{yx} = 12xy.$$

How do you find the ϕ ?

Trial:

$$\phi_x = 6xy^2 \quad (\text{take integral w/ respect to } x)$$

$$\phi = 3x^2y^2$$

$$\phi_y = 6x^2y + 1 \quad (\text{take integral w/ respect to } y)$$

$$\phi = 3x^2y^2 + ?$$

Solution:

$$\phi_x = 6xy^2$$

$$\phi = 3x^2y^2 + C(y)$$

$$\phi_y = 6x^2y + C'(y) \quad \left\{ \begin{array}{l} C'(y) = 1, \text{ so } C(y) = y + C_1 \end{array} \right.$$

$$\phi_y = 6x^2y + 1$$

$$\text{So, } \phi(x, y) = \boxed{3x^2y^2 + y} + \text{constant } C_1$$

Theorem: Vector field $F = (P, Q)$ is conservative if and only if

$$Q_x = P_y$$

$$P = 6xy^2 \quad \rightsquigarrow \quad P_y = 12xy \quad \left\{ \begin{array}{l} Q_x = P_y \end{array} \right.$$

$$Q = 6x^2y + 1 \quad \rightsquigarrow \quad Q_x = 12xy$$

$$P = y \quad \rightsquigarrow \quad P_y = 1 \quad \left\{ \begin{array}{l} Q_x \neq P_y \end{array} \right.$$

$$Q = -x \quad \rightsquigarrow \quad Q_x = -1$$

Ex: Check if the vector field

$$F(x, y) = (ye^{xy} + \cos y, xe^{xy} - x \sin y)$$

is conservative. If it is, find a potential function.

$$P = ye^{xy} + \cos y$$

$$Q = xe^{xy} - x \sin y$$

Trial:

$$P_y = u'v + v'u = e^{xy} + e^{xy}xy - \sin y$$

$$u = y \quad u' = 1$$

$$v = e^{xy} \quad v' = xe^{xy}$$

$$Q_x = u'v + v'u = e^{xy} + e^{xy}xy - (x \sin y) + x(\cos y)$$

$$u = x \quad u' = 1$$

$$v = e^{xy} \quad v' = ye^{xy}$$

$$u = x \quad u' = 1$$

$$v = \sin y \quad v' = \cos y$$

$$P_y = ye^{xy} + \cos y$$

$$P_y = e^{xy} - \sin y \quad (?) \quad X = e^{xy} + e^{xy}xy - \sin y$$

$$Q_x = e^{xy} - \sin y \quad (?) \quad X = e^{xy} + e^{xy}xy - \sin y$$

$$\phi_x = ye^{xy} + \cos y$$

$$\phi = e^{xy} + x \cos y + C(y)$$

$$\phi_y = xe^{xy} - x \sin y + C'(y)$$

$$\text{So, } C'(y) = 0$$

$$\text{So, } C(y) = C_1$$

$$\text{So, } \phi = e^{xy} + x \cos y + C_1$$

$$\phi = e^{xy} + x \cos y$$

$$\phi_y = xe^{xy} - x \sin y$$

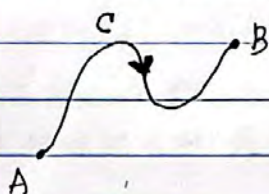
should be match...

Recall: Fundamental Theorem of Calculus.

$$\int f'(x) dx = f(x) + C$$

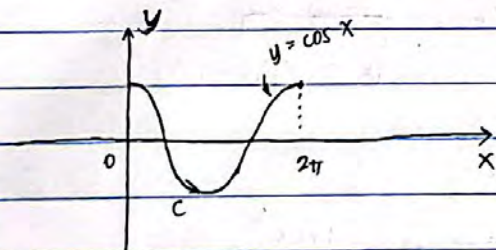
$$\int_a^b f'(x) dx = f(b) - f(a) = f(x) \Big|_a^b$$

Fundamental Theorem of Calculus for Multivariable functions:



$$\int_C \nabla F \cdot dr = \int_C \nabla \phi \cdot dr = \phi(B) - \phi(A)$$

Ex: Let $F(x, y) = (y, x)$, and C be the curve



Find $\int_C F \cdot dr$

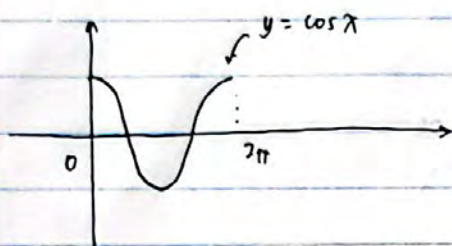
$$\nabla \phi = xy$$

$$= \int_C \nabla \phi \cdot dr = \int_0^{2\pi} xy \cdot dr = \int_0^{2\pi} x \cos x \cdot dx$$

June 24, 2025

Math 314

Ex: $F(x, y) = (y, x)$



$$\int_C F \cdot dr =$$

$$r(t) = (t, \cos(t))$$

$$F \cdot r'(t) = \overset{\cos(t)}{y} \cdot (1, -\sin(t))$$

$$dr = r'(t) dt$$

$$= y + (-x \sin(t)) = \cos(t) - t \sin(t)$$

$$0 \leq t \leq 2\pi \quad r'(t) = (1, -\sin(t))$$

$$\int_a^b F \cdot r'(t) dt = \int_0^{2\pi} y - x \sin(t) dt = yt + x \cos(t) \Big|_0^{2\pi}$$

$$= \int_0^{2\pi} \cos(t) - t \sin(t) dt$$

$$u = \cos(t)$$

$$du = u' dt$$

$$du = -\sin(t) dt$$

$$\int uv' = uv - \int v u'$$

$$= t \cos t - \int -\sin(t)$$

$$= t \cos t + \sin(t)$$

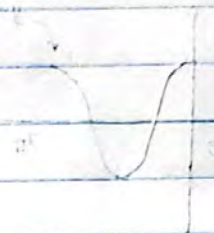
t	0	2\pi
u	1	1

$$x = t \quad y = \cos t$$

$$r(t) = (t, \cos(t)), \quad 0 \leq t \leq 2\pi$$

$$\int_C F \cdot dr = \int_0^{2\pi} F \cdot r'(dt) = \int_0^{2\pi} (y, x) \cdot (1, -\sin(t)) dt \quad (x, y) = (t, \cos(t))$$

$$= \int_0^{2\pi} (\cos t - t \sin(t)) dt = t \cos t \Big|_0^{2\pi} = 2\pi$$



$$F(x, y) = (y, x)$$

$$\begin{aligned} P = y &\rightarrow P_y = 1 \\ Q = x &\rightarrow Q_x = 1 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{conservative}$$

Find potential function $\phi(x, y)$.

$$F = \nabla \phi$$

$$(y, x) = (\phi_x, \phi_y)$$

$$\phi_x = y$$

$$\phi_y = x \rightarrow \phi = xy + C(x)$$

$$\phi_x = y + C'(x)$$

$$\text{So, } \phi(x, y) = xy.$$

$$\int_C F \cdot dr = \int_C \nabla \phi \cdot dr$$

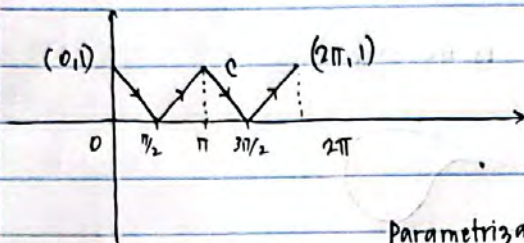
$$= \phi(\text{end point}) - \phi(\text{starting point})$$

$$= \phi(2\pi, 1) - \phi(0, 1)$$

$$= 2\pi(1) - 0(1)$$

$$= 2\pi$$

Ex: $F(x, y) = (y, x)$



Parametrization

$$\int_C F \cdot dr = ? = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

Method 1: Parametrize C_1, C_2, C_3, C_4 .

Then compute $\int_{C_1} F \cdot dr, \dots, \int_{C_4} F \cdot dr$

Then sum them.

Method 2: $\int_C F \cdot dr = \int_C \nabla \phi \cdot dr$

$$= \phi(\text{end point}) - \phi(\text{starting point})$$

$$= \phi(2\pi, 1) - \phi(0, 1)$$

$$= 2\pi$$

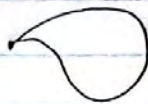
Theorem: If F is a conservative vector field then $\int_C F \cdot dr$ only depends on the starting point and the end point of C .

$$F(x,y) = (y, -x)$$

Curve C is closed if the starting point is the same as the end point.

* A closed curve is also called a loop.

Curve C is simple if it doesn't intersect itself.



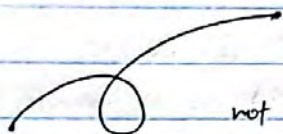
closed, simple



simple, not closed

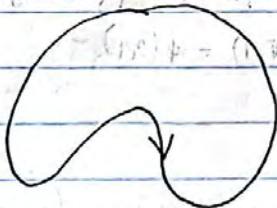


closed
not simple



not simple,
not closed

A simple, closed curve has an orientation:



Imagine that you walk along the curve.

If the bounded region is on your left.

the curve is positively oriented. Otherwise,

it is negatively oriented.

June 25, 2025

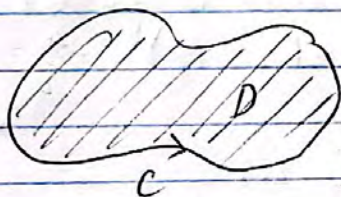
Green's Theorem :

Let $F = (P, Q)$ be a vector field,

C be a simple and closed curve.

$$\text{Then } \int_C F \cdot dr = \pm \iint_D (Q_x - P_y) dA$$

Sign depends/determined
on/by orientation of C .



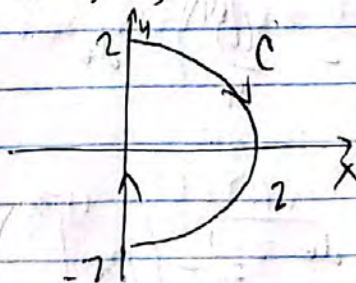
* D is the region bounded or enclosed by C .

* Green's theorem converts line integral
to double integral.

Green's theorem is consistent with the
Fundamental Theorem of Calculus
when F is conservative.

Ex: Find $\int_C (-y^3 + x) dx + (x^3 + y) dy$

where C is the curve:



$$P = -y^3 + x$$

$$Q = x^3 + y$$

$$P_y = \frac{\partial}{\partial y}(-y^3 + x) = -3y^2$$

$$Q_x = \frac{\partial}{\partial x}(x^3 + y) = 3x^2$$

$$= - \iint_D \left(\frac{x^4}{4} + xy - \left(\frac{y^4}{4} + xy \right) \right) dA$$

$$= - \iint_D 3x^2 + 3y^2 dA = 3(x^2 + y^2)$$

$$= \begin{aligned} &0 \leq r \leq 2 \\ &-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{aligned}$$

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 3(r^2) r dr d\theta = - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 3r^3 dr d\theta = - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[3 \frac{r^4}{4} \right]_0^2 d\theta$$

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[3r^2 \right]_0^2 d\theta$$

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12 d\theta = - \left[12\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = - \left(12 \left(\frac{\pi}{2} \right) - 12 \left(-\frac{\pi}{2} \right) \right) = -6\pi - 6\pi = -12\pi$$

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3(2)^4}{4} - \frac{3(0)^4}{4} \right) d\theta$$

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12 d\theta = - \left(12\theta \right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = - \left(12 \left(\frac{\pi}{2} \right) - 12 \left(-\frac{\pi}{2} \right) \right)$$

$$= 6\pi + 6\pi = 12\pi$$