Conditional probability

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<u>A motivating example:</u>
Suppose that a new family just moved to your neighborhood. You don't know anything about them besides the fact that they have two children. You are interested in knowing the gender of their children. Without further knowledge, the sample space is $S = \{BB, BG, GB, GG\}$.
The probability that both of them are boys is $1/4$. We write $P(BB) = 1/4$.

Suppose that you know that one of the children is a boy. Then the sample space is $S_1 = \{BB, BG, GB\}$.
Now the probability that both of them are boys is $1/3$. We write $P(BB|one\ is\ B) = 1/3$.

Suppose, instead, that you know that one of the children is a girl. Then the sample space is $S_2 = \emptyset$. The probability that both of them are boys is 0. We write $P(BB|one\ is\ G) = 0$.

This new kind of probability is called conditional probability. Knowing that some event already happened narrows down the sample space, thus changing the probability that another event occurs.

<u>Another example:</u>
Suppose that you flip a coin 10 times. The chance of getting all Heads is

$$P(all\ Heads) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}.$$

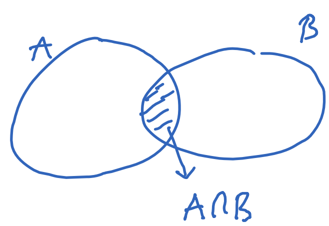
This is a very small probability. Now imagine that you already get 9 Heads in a row. You would think that you are due for a Tail, and that the chance of getting another Head is low because the chance of getting all 10 Heads is low. However,

$$P(all\ Heads|\ 9\ Heads) = P(the\ last\ flip\ is\ Head) = \frac{1}{2}.$$

In other words, you still have 50% chance of getting Head in the last flip.

When an event B is known to have occurred, the sample space (the set of all possible outcomes) is narrowed down to B. The probability that A occurs knowing that B occurred is measured by the ratio of $P(A \cap B)$ to $P(B)$. Thus,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Independent events:
Two events are called independent of each other if knowing that one event happened doesn't change the likelihood that the other event happens. For example:

A = event that a particular student shows up to class (class time is 11 AM - 12:20 PM).

B = event that it rains heavily from 10:30 to 11:30 AM.

Most likely, A and B are dependent on each other: $P(A|B) < P(A)$.

Consider another example: you flip a coin twice.

A = event that you get Head on the first flip

B = event that you get Head on the second flip

Whether *A* occurs or not has no effect on the likelihood that *B* occurs. That is, $P(B|A) = P(B)$.
This is exactly how independence of events are defined.

A and *B* are independent $\Leftrightarrow$ $P(B|A) = P(B) \Leftrightarrow P(A|B) = P(A) \Leftrightarrow P(A \text{ and } B) = P(A)P(B)$
If *A* and *B* are not independent, they are said to be **associated**.

Application:

You flip a coin 10 times. What is the probability of getting all Heads?
There are too many possible outcomes to list. However, you can take advantage of the fact that the results of all flips are independent of one another.

$$P(10\ H) = P(H \text{ and } 9\ H) = P(H)P(9\ H) = P(H)P(H \text{ and } 8\ H)$$
$$= P(H)P(H)P(8\ H) = \cdots = P(H)P(H)\ldots P(H) = P(H)^{10} = \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}} = \frac{1}{1024}.$$