Solution to Prob. 9 and 15 of HW2

9) $$f(x) = \frac{x^2 - x - 12}{x^2 + x - 6}$$

Domain:

For f to be well-defined, the denominator has to be nonzero.

$$x^2 + x - 6 = (x-2)(x+3)$$

For this to be nonzero, $x \neq 2$ and $x \neq -3$. Therefore, the domain of f is $\mathbb{R} \setminus \{-3, 2\}$, or equivalently, $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

Vertical asymptotes

we need to simplify the fraction first:

$$f(x) = \frac{x^2 - x - 12}{x^2 + x - 6} = \frac{(x+3)(x-4)}{(x-2)(x+3)} = \frac{x-4}{x-2}$$

The vertical asymptote is the line $x = 2$.

Holes:

Because the function is undefined at $x = -3$, but $x = -3$ is not a vertical asymptote, so the graph has a hole when $x = -3$. To determine the y-coordinate of the hole, we put $x = -3$ into the simplied $f(x)$:

$$\frac{(-3) - 4}{(-3) - 2} = \frac{-7}{-5} = \frac{7}{5}$$

Therefore, the hole on the graph has coordinate $(-3, 7/5)$.

Horizontal asymptotes:

$$f(x) = \frac{x^2 - x - 12}{x^2 + x - 6}$$

The numerator and the denominator are polynomials of the same degree. Only the highest powers matter when x is large.

$$f(x) \approx \frac{x^2}{x^2} = 1 \quad \text{when } x \text{ is large (toward } \infty \text{ or } -\infty)$$

Thus, $y = 1$ is the only horizontal asymptote.

(5) $$f(x) = \frac{-5x^4 - 3x^3 + x^2 - 10}{x^3 - 3x^2 + 3x - 1}$$

Domain:

For $f(x)$ to be well-defined, the denominator has to be nonzero. To find the values of x that make the denominator zero, we need to factor it.

Let $g(x) = x^3 - 3x^2 + 3x - 1$.

We see that $x=1$ is a root of g, so $x-1$ is a factor of $g(x)$. Synthetic division:

$$\begin{array}{c|cccc} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$$g(x) = (x-1)\underbrace{(x^2 - 2x + 1)}_{(x-1)(x-1)} = (x-1)^3$$

The only root of g is $x=1$. Therefore, the domain of f is $\mathbb{R} \setminus \{1\}$, or equivalently $(-\infty, 1) \cup (1, \infty)$.

Vertical asymptotes

We need to simplify the fraction first. The numerator $h(x) = -5x^4 - 3x^2 + x^2 - 10$ doesn't have root $x=1$, so it doesn't have a factor $x-1$. That means there are no common factors between the numerator and denominator. The fraction is already in a simplified form.

The vertical asymptote is the line $x=1$.

Holes

Because there are no common factors between the numerator and the denominator, the graph has no holes.

Horizontal asymptotes

Because the degree of the numerator is larger than the degree of the denominator, there are no horizontal asymptotes.

Slant asymptotes

Long division:

$$
\begin{array}{r l}
 & \quad -5x - 18 \\
x^3 - 3x^2 + 3x - 1 & \overline{\big)\, -5x^4 - 3x^3 + x^2 + 0x - 10} \\
 & -\;\; -5x^4 + 15x^3 - 15x^2 - 5x \\
\hline
 & \qquad -18x^3 + 16x^2 + 5x - 10 \\
 & \quad -\;\; -18x^3 + 54x^2 - 72x + 18 \\
\hline
 & \qquad\qquad -38x^2 + 77x - 28
\end{array}
$$

Thus,

$$f(x) = -5x - 18 + \underbrace{\frac{-38x^2 + 77x - 28}{x^3 - 3x^2 + 3x - 1}}_{\substack{\text{small when } x \text{ large because the} \\ \text{numerator has a smaller degree than} \\ \text{the denominator.}}}$$

Thus, $f(x) \approx -5x - 18$ when x is large.

The only slant asymptote of the graph of f is $y = -5x - 18$.