# Solution to Problem 6 and 20 of HW4

6) $$\frac{-x^3+4x}{x^2-9} = 4x \qquad (1)$$

* Requirement: $x^2-9 \neq 0$

Note that $x^2-9 = (x-3)(x+3)$.

The requirement is equivalent to $x \neq -3$ and $x \neq 3$.

* Multiply both sides of (1) by $x^2-9$:

$$-x^3+4x = 4x(x^2-9)$$

$$\rightsquigarrow \quad -x^3+4x = 4x^3-36x$$

$$\rightsquigarrow \quad 5x^3-40x = 0$$

$$\rightsquigarrow \quad 5x(x^2-8) = 0$$

$$\rightsquigarrow \quad x=0 \quad \text{or} \quad x^2=8$$

$$\rightsquigarrow \quad x=0 \quad \text{or} \quad x=\sqrt{8} \quad \text{or} \quad x=-\sqrt{8}$$

These values satisfy the requirement that $x \neq \pm 3$.

Therefore, the equation has three solutions: $x=0, \pm\sqrt{8}$.

20) $$\frac{5x^3-12x^2+9x+10}{x^2-1} \geq 3x-1$$

This is equivalent to

$$\frac{5x^3-12x^2+9x+10}{x^2-1} - (3x-1) \geq 0$$

$$\rightsquigarrow \quad \frac{5x^3-12x^2+9x+10}{x^2-1} - \frac{(3x-1)(x^2-1)}{x^2-1} \geq 0$$

$$\rightsquigarrow \quad \frac{5x^3-12x^2+9x+10}{x^2-1} - \frac{3x^3-x^2-3x+1}{x^2-1} \geq 0$$

$$\rightsquigarrow \quad \frac{5x^3-12x^2+9x+10-(3x^3-x^2-3x+1)}{x^2-1} \geq 0$$

$$\rightsquigarrow \quad \frac{2x^3-11x^2+12x+9}{x^2-1} \geq 0$$

Now we need to factor the numerator and the denominator.
The denominator is $x^2-1 = (x-1)(x+1)$.
The numerator $f(x) = 2x^3-11x^2+12x+9$ has a root $x=3$. Thus, $x-3$ is a factor of its. Synthetic division:

$$\begin{array}{c|cccc} 3 & 2 & -11 & 12 & 9 \\ & \downarrow & 6 & -15 & -9 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

$f(x) = (x-3)(2x^2-5x-3)$
To find the roots of $2x^2-5x-3$, we use the quadratic formula:

$$x = \frac{5 \pm \sqrt{(-5)^2-4(2)(-3)}}{4} = \frac{5 \pm \sqrt{25+24}}{4} = \frac{5 \pm 7}{4}$$

$$= -\tfrac{1}{2} \text{ or } 3$$

Thus, $2x^2-5x-3 = 2\left(x-\left(-\frac{1}{2}\right)\right)(x-3)$

Therefore, $f(x) = 2\left(x-\left(-\frac{1}{2}\right)\right)(x-3)^2$.
We rewrite the fraction as

$$\underbrace{\frac{2\left(x-\left(-\frac{1}{2}\right)\right)(x-3)^2}{(x-1)(x+1)}}_{g(x)} \geq 0$$

The special values are $-\frac{1}{2}, -1, 1, 3$.

Sign chart:

| $x$ | | $-1$ | | $-\frac{1}{2}$ | | $1$ | | $3$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x-(-\frac{1}{2})$ | $-$ | | $-$ | $0$ | $+$ | | $+$ | | $+$ |
| $(x-3)^2$ | $+$ | | $+$ | | $+$ | | $+$ | $0$ | $+$ |
| $x-1$ | $-$ | | $-$ | | $-$ | | $+$ | | $+$ |
| $x+1$ | $-$ | | $+$ | | $+$ | | $+$ | | $+$ |
| $g(x)$ | $-$ | | $+$ | $0$ | $-$ | | $+$ | $0$ | $+$ |

For $g(x)$ to be $\geq 0$, we need $-1 < x \leq -\frac{1}{2}$ or $x > 1$.
Therefore, the solutions to the inequality are $x \in (-1, -\frac{1}{2}] \cup (1, \infty)$.