Quadratic formula

Real and complex roots

Quadratic formula

Solve the equation

$$ax^2 + bx + c = 0$$

Methods:

- Completing the square
- Factoring
- Quadratic formula: $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

Example 1

Solve $2x^2 + 9x + 10 = 0$

Complete the square:

$$2\left(x^2 + \frac{9}{2}x + 5\right) = 0$$

$$\boxed{x^2 + \frac{9}{2}x + 5 = 0}$$

$$\underbrace{(x+d)^2} = (x+d)(x+d) = \underbrace{x^2 + \boxed{2dx} + d^2}$$

$$2dx = \frac{9}{2}x$$

$$d = \frac{9}{4}$$

$$\left(x + \frac{9}{4}\right)^2 - \left(\frac{9}{4}\right)^2 + 5 \quad = x^2 + \frac{9}{2}x + 5 = 0$$

$$\left(x + \frac{9}{4}\right)^2 - \underbrace{\frac{81}{16} + 5} = 0$$

$$\left(x + \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{80}{16} = 0$$

$$\left(x + \frac{9}{4}\right)^2 - \frac{1}{16} = 0$$

$$\left(x + \frac{9}{4}\right)^2 = \frac{1}{16} = \left(\frac{1}{4}\right)^2$$

$$x + \frac{9}{4} = \pm \frac{1}{4}$$

$$+: \quad x + \frac{9}{4} = \frac{1}{4}$$

$$-: \quad x + \frac{9}{4} = -\frac{1}{4}$$

$$x = -2, \quad x = -\frac{5}{2}$$

Example 1

Solve $2x^2+9x+10=0$

$x=\frac{p}{q}$: guess

p is a divisor of 10

q is a divisor of 2, $q>0$

$p=\pm1, \pm2, \pm5, \pm10$

$q=1, 2$

$\frac{p}{q}=\pm1, \pm2, \pm5, \pm10, \pm\frac{1}{2}, \cancel{\pm1}, \pm\frac{5}{2}, \cancel{\pm5}$

Factoring method:

$2(-2)^2+9(-2)+10=8-18+10=0$

$x=-2$ ✓

$\underbrace{2x^2+9x+10}$ divisible by $\underbrace{x-(-2)}_{x+2}$

synthetic division

$2x^2+9x+10 = 2\underbrace{\left(x^2+\frac{9}{2}x+5\right)}$

$$\begin{array}{r|rrr} -2 & 1 & \frac{9}{2} & 5 \\ & & -2 & -5 \\ \hline & 1 & \frac{5}{2} & 0 \end{array}$$

$x^2+\frac{9}{2}x+5=\underbrace{(x+2)}_{x=-2}\underbrace{\left(x+\frac{5}{2}\right)}_{x=-5/2}=0$

Example 1

Quadratic formula $ax^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Solve $2x^2+9x+10=0$

$a=2 \quad b=9 \quad c=10$

$\Delta = b^2-4ac$: discriminant of the quadratic polynomial.

$$b^2-4ac = 9^2-4(2)(10) = 81-80 = 1$$

If $\Delta=-1$

$i=\sqrt{-1}$ imaginary unit

$i^2=-1$

$$x=\frac{-9\pm\sqrt{1}}{2(2)}=\frac{-9\pm 1}{4}$$

plus : $x=\frac{-9+1}{4}=\frac{-8}{4}=-2$

minus : $x=\frac{-9-1}{4}=\frac{-10}{4}=\frac{-5}{2}$

Arithmetic of Complex numbers

Complex numbers: $\underbrace{a}_{\text{real part}}+\underbrace{b}_{\text{imaginary part}}i$ where a and b are real numbers.

Fund. thm of alg: every poly. has a complex root.

Rule: $i^2=-1$

x^2+1 $\quad x=i \quad i^2+1=-1+1=0$

$x=-i \quad (-i)^2+1$

$=(-i)(-i)+1$

$=i^2+1=0$

Example: $x=1-2i,\ y=2+i$

$a=1\ \ b=-2 \qquad a=2\ \ b=1$

$2=\underbrace{2}_{a}+\underbrace{0}_{b}i$

$$\begin{array}{r} x=1-2i \\ +\ \ y=2+i \\ \hline x+y=3-i \end{array}$$

Arithmetic of Complex numbers

Rule: $i^2 = -1$

Example: $x = 1 - 2i, \;\; y = 2 + i$

$$xy = (1-2i)(2+i)$$
$$= 2 + (-2i)2 + i + (-2i)i$$
$$= 2 - 4i + i - 2\underbrace{i^2}_{-1}$$
$$= 2 - 4i + i + 2 = 4 - 3i$$

standard form

$$\frac{x}{y} = \frac{1-2i}{2+i} = \frac{(1-2i)(2-i)}{(2+i)(2-i)}$$

$a - bi$ is the conjugate of $a + bi$

$$(a-bi)(a+bi) = a^2 - \cancel{bai} + \cancel{abi} - b^2\underbrace{i^2}_{-1} = \underline{a^2 + b^2}$$

Arithmetic of Complex numbers

Rule: $i^2 = -1$

Example: $x = 1 - 2i, \;\; y = 2 + i$

$$\frac{x}{y} = \frac{(1-2i)(2-i)}{(2+i)(2-i)} = \frac{2 - 4i - i + (-2i)(-i)}{2^2 + 1^2} = \frac{2 - 5i + 2i^2}{5} = \frac{2 - 5i - 2}{5} = -i$$

$$\frac{1-2i}{2+i} = -i$$

$$-i = 0 + (-1)i$$

Find real and complex roots of a polynomial

Example 2: $x^3 + 6x^2 + 6x + 5 = 0$

cubic equation

guess root $x = p/q$

Rational roots test

$p = \pm 1, \pm 5$

$q = 1$

$\frac{p}{q} = \pm 1, \pm 5$

$x = -5$ is a root

synthetic division

$$\begin{array}{r|rrrr} -5 & 1 & 6 & 6 & 5 \\ & \downarrow & -5 & -5 & -5 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

$x^3 + 6x^2 + 6x + 5 = (x+5)(x^2 + x + 1)$

Find real and complex roots of a polynomial

Example 2: $x^3 + 6x^2 + 6x + 5 = 0$

$$\underbrace{(x+5)}_{0}\underbrace{(x^2+x+1)}_{0} = 0$$

or

$x = -5$ is a root

$x^2 + x + 1 = 0$ quadratic formula

$a=1 \quad b=1 \quad c=1$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$b^2 - 4ac = 1^2 - 4(1)(1) = 1-4 = -3$

$$x = \frac{-1 \pm \sqrt{-3}}{2}$$

$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3 i^2} = \pm\sqrt{3}\, i$

$$x = \frac{-1 \pm \pm\sqrt{3}\, i}{2} = \frac{-1 \pm \sqrt{3}\, i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Cubic eq. has three roots:

$x = -5, \quad x = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad x = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

Find real and complex roots of a polynomial

Example 2: $\underbrace{x^3 + 6x^2 + 6x + 5}_{} = 0$

$$= (x-(-5))\left(x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)$$

$$= (x+5)\underbrace{\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}_{}$$