Exponential equations and inequalities

Exponential equations

$\ln$

$\log$

$$a^b = c \Leftrightarrow b = \log_a c$$

- **Example 1:** Solve the equation $4^x = 8$

$$x = \log_4 8 = \frac{\log_e 8}{\log_e 4} = \frac{\ln 8}{\ln 4} = \frac{\ln(2^3)}{\ln(2^2)} = \frac{3\cancel{\ln 2}}{2\cancel{\ln 2}} = \frac{3}{2}$$

$$\boxed{x = \frac{3}{2}}$$

Exponential equations

x^4 : power function

4^x : exponential function

- **Example 2:** Solve the equation $\underbrace{x^4 = 8}_{\text{power eq.}}$

$$x^4 = 8 \longrightarrow x = \pm\sqrt[4]{8} = \pm 8^{1/4} \approx \pm 1.6818$$

Exponential equations

$$\overset{e}{a}^{\overset{-2x}{b}} = \overset{4/5}{c} \Leftrightarrow b = \log_a c$$

- **Example 3:** Solve the equation $3 + 5e^{-2x} = 7$

$5e^{-2x} = 7 - 3 = 4$

$e^{-2x} = \frac{4}{5}$

$-2x = \log_e\left(\frac{4}{5}\right) = \ln\left(\frac{4}{5}\right)$

$x = \frac{\ln(4/5)}{-2}$

Exponential equations

$$a^b = c \Leftrightarrow b = \log_a c$$

- **Example 4:** Solve the equation $\underbrace{3^x} + 2(\underbrace{3^{-x}}) = 3$

$3^{-x} = \frac{1}{3^x}$

$3^x + 2\frac{1}{3^x} = 3$

$\downarrow$

Let $t = 3^x$.

$\left(t + 2\frac{1}{t} = 3\right) \times t$

Mult. by t:

$t^2 + 2 = 3t \longrightarrow t^2 - 3t + 2 = 0$

$\underline{t = 1}$ or $\underline{t = 2}$

If $t=1$: $3^x = 1 \longrightarrow x = \log_3 1 = 0$

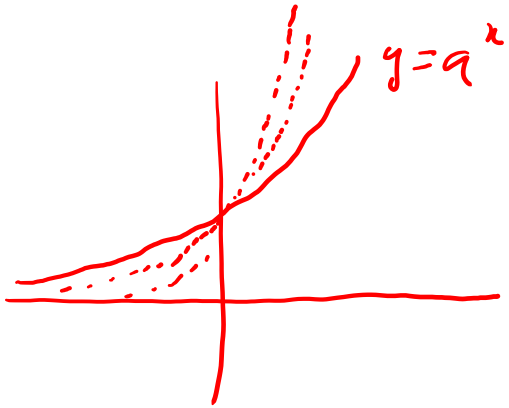
If $t=2$: $3^x = 2 \longrightarrow x = \log_3 2 = \dfrac{\ln 2}{\ln 3} \approx 0.6309$

Conclusion! two solutions

$$x = 0, \quad x = \frac{\ln 2}{\ln 3}$$

Exponential inequalities

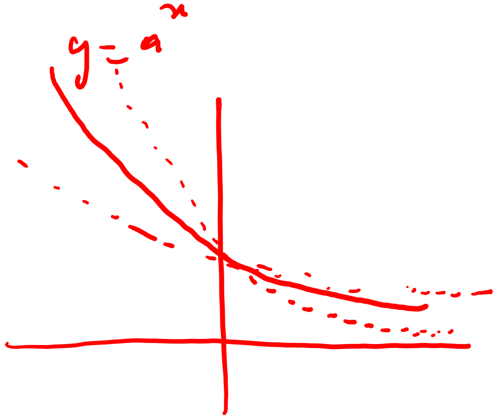
a^x increasing



- If $a > 1$,

$$a^b > c \Leftrightarrow b > \log_a c$$

- If $a < 1$,



$$a^b > c \Leftrightarrow b < \log_a c$$

Exponential inequalities

- **Example 5:** Solve the inequality $4\left(\frac{2}{3}\right)^x - 1 < 2$

Add 1: $4\left(\frac{2}{3}\right)^x < 3$

Div. 4 : $\left(\frac{2}{3}\right)^x < \frac{3}{4}$ $a = \frac{2}{3} < 1,\ b = x\ ,\ c = \frac{3}{4}$

$x > \log_{\frac{2}{3}}\left(\frac{3}{4}\right) = \frac{\ln\left(\frac{3}{4}\right)}{\ln\left(\frac{2}{3}\right)}$

Conclusion: $x \in \left(\frac{\ln(3/4)}{\ln(2/3)}, \infty\right)$

Exponential inequalities

- **Example 6:** Solve the inequality $4 - 5e^{-x} \geq 2$

Sub. 4: $-5e^{-x} \geq -2$

$e \approx 2.71\ldots > 1$

Mult. -1: $5e^{-x} \leq 2$

Div. 5: $e^{-x} \leq \frac{2}{5}$

$a = e, \; b = -x, \; c = \frac{2}{5}$

$-x \leq \log_e\left(\frac{2}{5}\right) = \ln\left(\frac{2}{5}\right)$

$a^b \leq c$

$b \leq \log_a c$

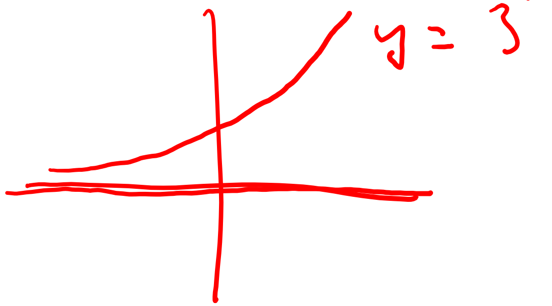
Mult. -1:

$x \geq -\ln\left(\frac{2}{5}\right)$

$\boxed{x \in \left[-\ln\left(\frac{2}{5}\right), \infty\right)}$

Exponential inequalities

- **Example 7:** Solve the inequality $3^x - 2(\underbrace{3^{-x}}_{\frac{1}{3^x}}) > 1$

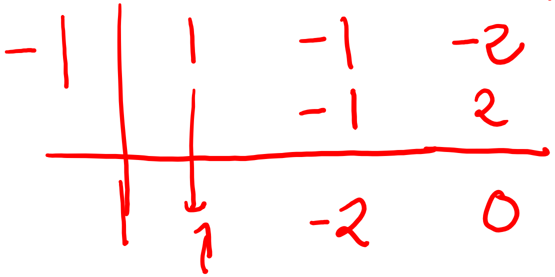


Let $t = 3^x > 0$

$$\left(t - 2\frac{1}{t} > 1\right) \times t$$

$$t^2 - 2 > t$$

$$t^2 - t - 2 > 0$$



$$\underbrace{(t+1)}_{>0}(t-2) > 0 \rightarrow t-2 > 0 \rightarrow t > 2 \rightarrow 3^x > 2 \rightarrow x > \log_3 2$$

conclusion: $x \in (\log_3 2, \infty)$

Exponential inequalities

- **Example 8:** Solve the inequality $3^x + 2(3^{-x}) > 3$

Let $t = 3^x > 0$

$$\left(t + 2\frac{1}{t} > 3\right) \times t$$

$$t^2 + 2 > 3t$$

$$t^2 + 2 - 3t > 0$$

$$(t-1)(t-2) > 0$$

t	0		1		2	
$t-1$	$-$		0	$+$		$+$
$t-2$	$-$			$-$	0	$+$
$(t-1)(t-2)$	$+$		0	$-$	0	$+$

$t < 1$ or $t > 2$

$t < 1$ or $t > 2$

$3^x < 1$ or $3^x > 2$

$x < \log_3 1 = 0$ or $x > \log_3 2 = \dfrac{\ln 2}{\ln 3}$

Conclusion: $x < 0$ or $x > \dfrac{\ln 2}{\ln 3}$

$$\boxed{x \in (-\infty, 0) \cup \left(\frac{\ln 2}{\ln 3}, \infty\right)}$$