Rational functions

Domain, simplification, vertical and horizontal asymptotes

Rational functions

$$R(x) = \frac{P(x)}{Q(x)}$$ where P(x) and Q(x) are polynomials.

Examples:

$$\frac{2x+1}{x-1}, \quad \underbrace{x+1}_{\mathbb{R}}, \quad \frac{-2}{x^2-1}, \quad \frac{x^2}{x^3-2x+1}, \ \ldots.$$

Domain

(a) $\dfrac{2x+1}{x-1}$

domain $= (-\infty, 1) \cup (1, \infty)$

$= \mathbb{R} \setminus \{1\}$

(b) $x+1$

domain $= \mathbb{R} = (-\infty, \infty)$

(c) $\dfrac{x^2}{x^3-2x+1}$

everything, except the value of x s.t. $x^3-2x+1=0$

domain $= \mathbb{R} \setminus \left\{1, \dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}\right\}$

$x^3 - 2x + 1 = 0$

$x = 1$ is a root

$$
\begin{array}{c|cccc}
1 & 1 & 0 & -2 & 1 \\
 & \downarrow & 1 & 1 & -1 \\
\hline
 & 1 & 1 & -1 & 0
\end{array}
$$

$x^3 - 2x + 1 = (x-1)(x^2+x-1) = 0$

either $x = 1$ or $x^2 + x - 1 = 0$

$a = 1,\ b = 1,\ c = -1$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Simplification

$\frac{6}{8} = \frac{3}{4}$

To simplify a rational function is to cancel the common factors, if there are any, of the numerator and the denominator.

(a) $\dfrac{4x}{x^3 - 2x^2} = \dfrac{4x}{x^2(x-2)} = \dfrac{4\cancel{x}}{\cancel{x}x(x-2)} = \dfrac{4}{\underbrace{x(x-2)}_{\text{simplified form}}}$ ✓

Simplification

(b) $\dfrac{x^2 - x - 2}{x^2 - 3x + 2}$

$x^2 - x - 2 = (x+1)(x-2)$

$x = -1$ is a root

-1	1	-1	-2
		-1	2
	1	-2	0

$x^2 - 3x + 2 = (x-1)(x-2)$

$x = 1$ is a root

1	1	-3	2
		1	-2
	1	-2	0

$\dfrac{x^2 - x - 2}{x^2 - 3x + 2} = \dfrac{(x+1)\cancel{(x-2)}}{(x-1)\cancel{(x-2)}} = \dfrac{x+1}{x-1}$ Simplified form

Simplification

(b) $\underbrace{\dfrac{x^2 - x - 2}{x^2 - 3x + 2}}_{\text{domain} = \mathbb{R} \setminus \{1,2\}} = \dfrac{(x+1)(x-2)}{(x-1)(x-2)} = \dfrac{x+1}{\underbrace{x-1}_{\mathbb{R} \setminus \{1\}}}$

Vertical asymptotes

when $x \approx a$: $P(x) \approx P(a) \neq 0$, $Q(x) \approx Q(a) = 0$

The vertical line x = a is a vertical asymptote of R(x) = P(x)/Q(x) if

$R(x) = \frac{P(x)}{Q(x)}$

$$P(a) \neq 0 \qquad \text{and} \qquad Q(a) = 0$$

Examples:

(a) $$\frac{2}{x^2 + x - 2}$$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1 \Rightarrow$ vertical asymptote

or $x = -2$

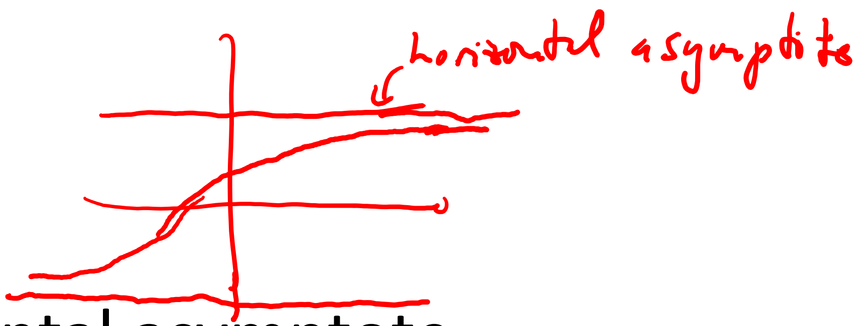
Vertical asymptotes

(b) $\dfrac{x^2+x-2}{x^2-3x+2}$ $\longrightarrow P(x) \neq 0$, $\longrightarrow Q(x) = 0$

$x^2 - 3x + 2 = 0$: $x = 1 \longrightarrow P(1) = 0$; $\boxed{x = 2} \longrightarrow P(2) = 4$

[Online graphing tool: https://www.geogebra.org/calculator]

Horizontal asymptotes



A rational function $R(x) = \frac{P(x)}{Q(x)}$ has a horizontal asymptote

only when the degree of P(x) is *less than or equal* to the degree of Q(x).

- If degree of P(x) is less than the degree of Q(x), then y = 0 is the only horizonal asymptote.

Horizontal asymptotes

- If P(x) and Q(x) have the same degree, then the only horizontal asymptote is

$$y = \frac{a}{b}$$

where a and b are the leading coefficients of P(x) and Q(x), respectively.

Examples:

$x \approx 1,000,000,000$

(a) $\frac{2x}{x^2 - 4}$ ≈ 0 when x large

Horizontal asymptotes

(b) $\dfrac{2-x}{1+x}$ $\underset{x \text{ large}}{\approx}$ $\dfrac{-x}{x} = -1$

$\dfrac{x^2 + x + 1}{-2x^2 - 3} \approx \dfrac{x^2}{-2x^2} = -\dfrac{1}{2}$

(c) $\dfrac{x^3 - x + 1}{x^2 + 4}$ no horizontal asymptote