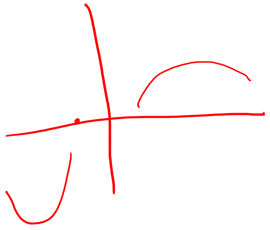
Rational functions

Sign chart, graph, slant asymptotes, polynomial asymptotes

Sign chart

$$R(x) = \frac{P(x)}{Q(x)}$$

\+ − 0



- Step 1: factor P(x) and Q(x) over the real numbers.
- Step 2: find all the real roots of P(x) and Q(x).
- Step 3: put these special numbers on a chart
- Step 4: determine the sign of the R(x) between two special numbers by plugging in a special value of x.

Sign chart

Example 1:

$x=1$

$$R(x)=\frac{x^2-3x+2}{x+5}$$

$$\begin{array}{r|rrr} 1 & 1 & -3 & 2 \\ & & 1 & -2 \\ \hline & 1 & -2 & 0 \end{array}$$

$(x-1)(x-2)$

- Factor the numerator and denominator:

$$R(x)=\frac{(x-1)(x-2)}{x+5}$$

- The special numbers are: -5, 1, 2

Sign chart

$$R(x) = \frac{(x-1)(x-2)}{x+5}$$

- Put the special numbers on a chart

x	-5	1	2
R(x)			

- Determine the signs

x		-5		1		2	
R(x)	-		+	0	-	0	+

Sign chart

$2^3 - 2 - 6 = 0$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -1 & -6 \\ & & 2 & 4 & 6 \\ \hline & 1 & 2 & 3 & 0 \end{array}$$

Example 2:

$x-2$

$$\mathrm{R(x)} = \frac{x^3 - x - 6}{x^2 + 3x + 2}$$

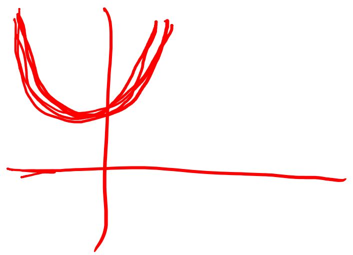
$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- Factor the numerator and denominator:

$a = 1 \quad b = 2 \quad c = 3$

$b^2 - 4ac = 4 - 4(1)(3) = -8 < 0$

$$\mathrm{R(x)} = \frac{\overset{-}{(x-2)}\overset{+}{(x^2 + 2x + 3)}}{\underset{-}{(x+1)}\underset{-}{(x+2)}}$$



Note that $x^2 + 2x + 3$ is always positive.

Sign chart

- The special numbers are -2, -1, 2.
- Put the special numbers on a chart:

x		-2		-1		2	
R(x)							

- Determine the signs:

x		-2		-1		2	
R(x)	-		+		-	0	+

Asymptotic behaviors

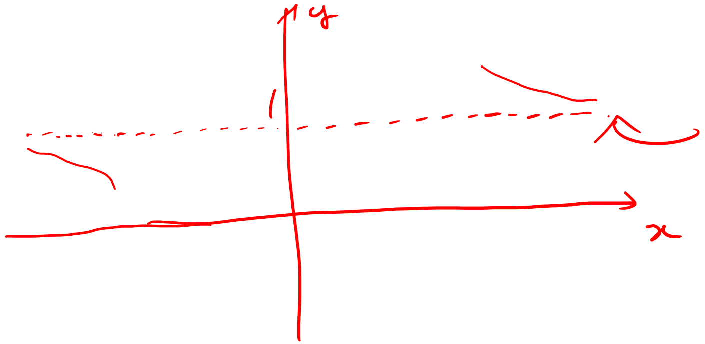
end behaviors

1 000 000 000

Question: What is R(x) close to when x is large (on the positive side or negative side)?

-1 000 000 000

<u>Example 1:</u> $$R(x) = \frac{x+1}{x-1} \approx \frac{x}{x} = 1$$



y=1 horizontal asymptote

Asymptotic behaviors

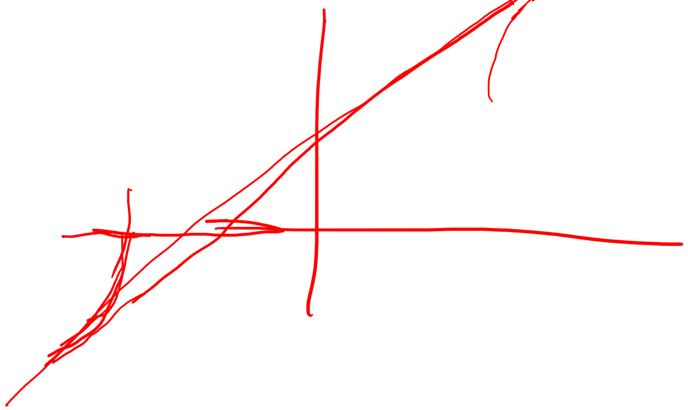
Question: What is R(x) close to when x is large (on the positive side or negative side)?

Example 2: $\mathrm{R}(x) = \dfrac{x^2}{x-1}$

$$\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ & & 1 & 1 \\ \hline & 1 & 1 & 1 \end{array}$$

$R(x) = x+1+\underbrace{\dfrac{1}{x-1}}_{\approx 0}$ when x large

$y = x+1$ slant asymptote



$R(x) \approx x+1$

Asymptotic behaviors

Question: What is R(x) close to when x is large (on the positive side or negative side)?

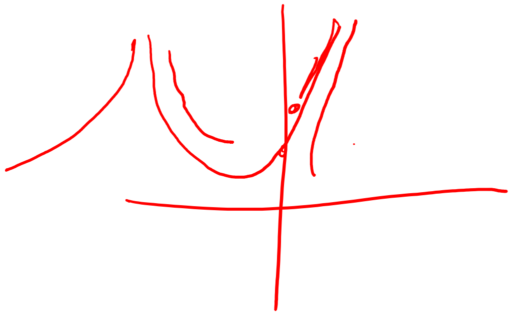
<u>Example 3:</u> $R(x) = \frac{x^3 - x^2 + 1}{x-2}$

$$\begin{array}{c|cccc} 2 & 1 & -1 & 0 & 1 \\ & & 2 & 2 & 4 \\ \hline & 1 & 1 & 2 & 5 \end{array}$$

$\underbrace{1 \quad 1 \quad 2}_{\text{quotient}} \quad \underbrace{5}_{\text{remainder}}$

$R(x) = x^2 + x + 2 + \underbrace{\frac{5}{x-2}}$

when x is large, ≈ 0, $R(x) \approx \underbrace{x^2 + x + 2}$



Graphing a rational function

- Step 1: determine the domain by factoring the denominator
- Step 2: factor the numerator and simplify the rational function
- Step 3: find the x-intercepts and the y-intercept
- Step 4: determine vertical asymptotes and holes
- Step 5: determine asymptotic behaviors (horizonal/slant/polynomial asymptotes)
- Step 6: make a sign diagram and sketch the graph

Graphing a rational function

Example 1:

$$R(x) = \frac{x^3 - x^2 - 2x}{x^2 - 3x + 2}$$

$= (x-1)(x-2)$

Step 1: factor the denominator to find the domain

$$\mathbb{R} \setminus \{1, 2\} = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$$

Graphing a rational function

Example 1:

$$R(x) = \frac{x^3 - x^2 - 2x}{(x-1)(x-2)} = \frac{x(x+1)(x-2)}{(x-1)(x-2)}$$

Step 2: factor the numerator to simplify

$x=2$

$x(\underbrace{x^2 - x - 2})$ $\quad$ $x=-1$ is a root

$$\begin{array}{r|rrr} -1 & 1 & -1 & -2 \\ & & -1 & 2 \\ \hline & 1 & -2 & 0 \end{array}$$

$x(x+1)(x-2)$

Graphing a rational function

Example 1:

$$R(x) = \frac{x\,(x+1)}{x-1}$$

<u>Step 3</u>: find the $\underbrace{\text{x-intercepts}}_{x=0 \text{ or } x=-1}$ and $\underbrace{\text{y-intercept}}_{R(0)=0}$

Graphing a rational function

Example 1:

$$R(x) = \frac{x\,(x+1)}{x-1}$$

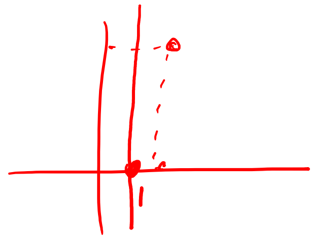
<u>Step 4</u>: find the vertical asymptotes and holes

vertical asymptotes: denominator = 0

$x = 1$ is a vertical asymptote

holes: both numerator & den. = 0

$x = 2$, $R(2) = 6$



Graphing a rational function

$$\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ & & 1 & 2 \\ \hline & 1 & 2 & 2 \end{array}$$

Example 1:

$$R(x) = \frac{x\,(x+1)}{x-1} = \frac{x^2+x}{x-1} = x+2+\underbrace{\frac{2}{x-1}}_{\approx 0 \text{ when } x \text{ large}}$$

Step 5: find the asymptotic behaviors

$$R(x) \approx \underbrace{x+2}_{\text{slant asymptote}}$$

Graphing a rational function

Example 1:

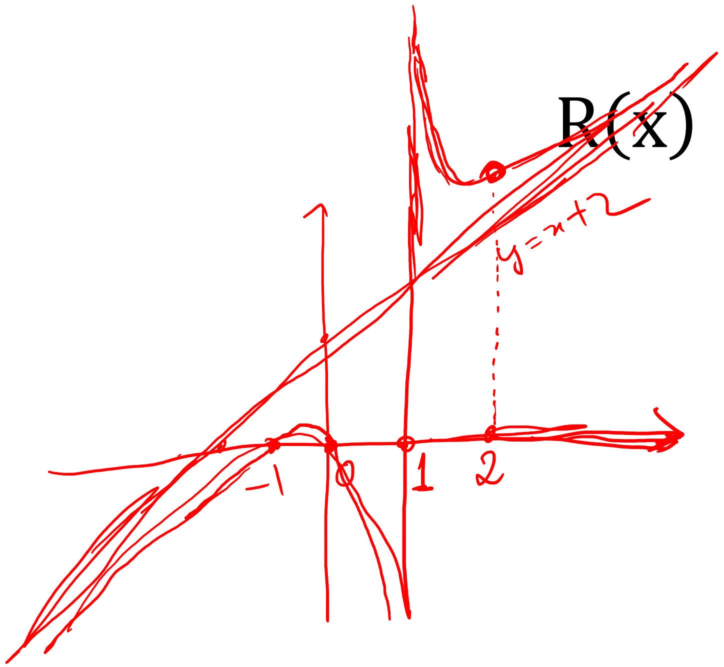
$$R(x) = \frac{x\,(x+1)}{x-1}$$

<u>Step 6</u>: sign chart

x		-1		0		1	
$R(x)$	$-$	0	$+$	0	$-$		$+$

Graphing a rational function

Example 1: (double check with GeoGebra.org)



$$R(x) = \frac{\overset{\approx 1(1+1)=2}{x\,(x+1)}}{\underset{}{x-1}} \quad \approx 0,\ <0$$

$x \approx 1,\ x > 1$

$x \approx 1,\ x < 1$

$$\frac{2}{-0.0001}$$

Graphing a rational function

Example 2:

$$R(x) = \frac{-x^3 + x^2 - 2}{x - 1}$$

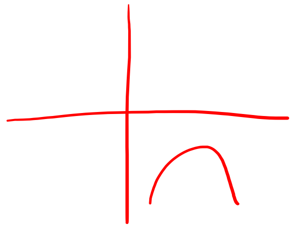
Step 1: factor the denominator to find the domain

$D = \mathbb{R} \setminus \{1\}$

Graphing a rational function

Example 2:

$$R(x) = \frac{-x^3 + x^2 - 2}{x - 1}$$



Step 2: factor the numerator to simplify

$$R(x) = \frac{(x+1)(\overbrace{-x^2+2x-2}^{-})}{x-1}$$

Graphing a rational function

Example 2:

$$R(x) = \frac{(x+1)(-x^2+2x-2)}{x-1}$$

<u>Step 3</u>: find the $\underbrace{\text{x-intercepts}}_{(x,y)=(-1,0)}$ and $\underbrace{\text{y-intercept}}_{(x,y)=(0,2)}$

Graphing a rational function

Example 2:

$$R(x) = \frac{(x+1)(-x^2+2x-2)}{x-1}$$

Step 4: find the vertical asymptotes and holes

$x = 1$

Graphing a rational function

Example 2:

$$\mathrm{R(x)} = \frac{-x^3 + x^2 - 2}{x - 1}$$

Step 5: find the asymptotic behaviors

$$R(x) = -x^2 - \underbrace{\frac{2}{x-1}}_{\approx 0 \text{ when } x \text{ large}} \approx -x^2 \quad \text{polynomial asymptote}$$

Graphing a rational function

Example 2:

$$R(x) = \frac{(x+1)\overbrace{(-x^2+2x-2)}^{-}}{x-1}$$

Step 6: sign chart

$x = -1$, $x = 1$

x		-1		1	
$R(x)$	$-$	0	$+$		$-$

Graphing a rational function

Example 2: (double check with GeoGebra.org)

$$R(x) = \frac{\overset{\approx 2}{(x+1)}\overset{\approx -1}{(-x^2+2x-2)}}{x-1} \quad \approx -2$$

$x - 1 \approx 0, < 0$

$-x^2$

$x \approx 1, x > 1$

$x \approx 1, x < 1$