Function composition

Review function operations

$f(x) = 2x$, $g(x) = x^2$

Given two functions f(x) and g(x), we learned in Math 111A:

- *Addition*: $(f + g)(x) = f(x) + g(x) = 2x + x^2$ $(f+g)(1) = 2(1) + 1^2 = 3$
- *Subtraction*: $(f - g)(x) = f(x) - g(x)$
- *Multiplication*: $(fg)(x) = f(x)g(x)$
- *Division*: $(f/g)(x) = \dfrac{f(x)}{g(x)}$

Another operation: composition

- *Recall the problem*:

Find the function f that takes a real number x and performs the following four steps: (1) add 2; (2) square; (3) subtract 1; (4) take the reciprocal.

$$x \xrightarrow{①} x+2 \xrightarrow{②} (x+2)^2 \xrightarrow{③} (x+2)^2-1 \xrightarrow{④} \frac{1}{(x+2)^2-1}$$

$$x \to \boxed{f} \to y \qquad x \to \boxed{\boxed{f} \to \boxed{g} \to \boxed{h} \to \boxed{k}} \to y$$

$\boxed{\text{composite function}}$

Composition

$$x \xrightarrow{f} \underbrace{x+2}_{f(x)} \xrightarrow{g} \underbrace{(x+2)^2}_{g(f(x))} \xrightarrow{h} \underbrace{(x+2)^2-1}_{h(g(f(x)))} \xrightarrow{k} \underbrace{\frac{1}{(x+2)^2-1}}_{\underbrace{k(h(g(f(x))))}_{(\underbrace{k \circ h \circ g \circ f}_{\text{Composite function}})(x)}}$$

$k \circ h \circ g \circ f$: k composed with h composed with g composed with f.

Composite functions

- Notations:

$f+g \quad f-g \quad fg \quad \frac{f}{g}$

Name: $f \circ g$

Evaluation: $(f \circ g)(x), = f(g(x))$

$f \to g \to h$

Name: $h \circ g \circ f$

Evaluation: $(h \circ g \circ f)(x), = h(g(f(x)))$

Composite functions

- **Example 1:** $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$

Find $f \circ g$ and $g \circ f$

$(f \circ g)(x) = f(\underbrace{g(x)})$

inside out : $f(\sqrt{x}) = \sqrt{x}^2 + 1 = x + 1$

outside in : $g(x)^2 + 1 = \sqrt{x}^2 + 1 = x + 1$

$$\boxed{(f \circ g)(x) = x + 1}$$

$(g \circ f)(x) = g(\underbrace{f(x)}) = g(x^2 + 1) = \sqrt{x^2 + 1}$

$$\boxed{(g \circ f)(x) = \sqrt{x^2 + 1}}$$

$f \circ g \neq g \circ f$.

Composition functions

- **Example 2:** $f(x) = 2x, \quad g(x) = \sqrt{x-5}, \quad h(x) = \dfrac{1}{x}$

Find $(f \circ f)(0.1)$ and $(f \circ g \circ h)(1)$

$f(f(0.1))$

$= f(0.2)$

$= 2 \times 0.2$

$= 0.4$

$h(1) = \frac{1}{1} = 1$

$g(1) = \sqrt{1-5} = \sqrt{-4}$ ✗

$(f \circ g \circ h)(1)$ not defined!

1 lies outside of the domain.

Composition functions

- **Example 2**: $f(x) = 2x, \quad g(x) = \sqrt{x-5}, \quad h(x) = \dfrac{1}{x}$

Find $f \circ g \circ h$ and its domain.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\frac{1}{x}\right)\right) = f\left(\sqrt{\frac{1}{x} - 5}\right)$$

$$= 2\sqrt{\frac{1}{x} - 5} = (f \circ g \circ h)(x)$$

Domain: $\quad x \neq 0, \quad \frac{1}{x} - 5 \geq 0$

Composition functions

- **Example 2**: $f(x) = 2x, \quad g(x) = \sqrt{x-5}, \quad h(x) = \dfrac{1}{x}$

Find $f \circ g \circ h$ and its domain.

$$\frac{1}{x} - 5 = \frac{1-5x}{x}$$

Sign chart:

x		0		$\frac{1}{5}$	
$1-5x$	$+$		$+$	0	$-$
domain	$-$		$+$	0	$-$

For $f \circ g \circ h$ to be well-defined

$$\begin{cases} x \neq 0 \\ \frac{1}{x} - 5 \geq 0 \end{cases} \iff \begin{cases} x \neq 0 \\ 0 \leq x \leq \frac{1}{5} \end{cases}$$

$$\iff 0 < x \leq \frac{1}{5}$$

$x \in (0, \frac{1}{5}]$.

The domain