

Function composition

Review function operations

$$f(x) = 2x, \quad g(x) = x^2$$

Given two functions $f(x)$ and $g(x)$, we learned in Math 111A:

- *Addition:* $(f + g)(x) = f(x) + g(x) = 2x + x^2$ $(f + g)(1) = 2(1) + 1^2 = 3$
- *Subtraction:* $(f - g)(x) = f(x) - g(x)$
- *Multiplication:* $(fg)(x) = f(x)g(x)$
- *Division:* $(f/g)(x) = \frac{f(x)}{g(x)}$

Another operation: composition

- *Recall the problem:*

Find the function f that takes a real number x and performs the following four steps: (1) add 2; (2) square; (3) subtract 1; (4) take the reciprocal.

$$x \xrightarrow{\textcircled{1}} x + 2 \xrightarrow{\textcircled{2}} \underbrace{(x + 2)^2}_{\textcircled{3}} \xrightarrow{\textcircled{4}} \frac{1}{\underbrace{(x + 2)^2 - 1}_{\textcircled{4}}}}$$

$$x \rightarrow \boxed{f} \rightarrow y$$

$$x \rightarrow \boxed{f} \rightarrow \boxed{g} \rightarrow \boxed{h} \rightarrow \boxed{k} \rightarrow y$$

Composite function

Composition

$$x \xrightarrow{f} \underbrace{x+2}_{f(x)} \xrightarrow{g} \underbrace{(x+2)^2}_{g(f(x))} \xrightarrow{h} \underbrace{(x+2)^2 - 1}_{h(g(f(x)))} \xrightarrow{k} \frac{1}{\underbrace{(x+2)^2 - 1}_{k(h(g(f(x))))}}$$

$(k \circ h \circ g \circ f)(x)$
Composite function

$k \circ h \circ g \circ f$: k composed with h
Composed with g
Composed with f .

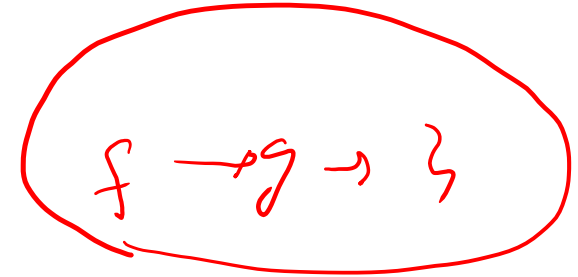
Composite functions

- Notations:

$$f+g \quad f-g \quad fg \quad \frac{f}{g}$$

Name: $f \circ g$

Evaluation: $(f \circ g)(x) = f(\underline{g(x)})$



Name: $h \circ g \circ f$

Evaluation: $(h \circ g \circ f)(x) = h(g(f(x)))$

Red arrows pointing from the innermost 'f' in the evaluation formula to the 'f' in the name, and from the 'g' to the 'g' in the name, and from the 'h' to the 'h' in the name.

Composite functions

- **Example 1:** $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$

Find $f \circ g$ and $g \circ f$

$$(f \circ g)(x) = f(g(x))$$

inside out:

$$f(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1$$

outside in:

$$g(x)^2 + 1 = (\sqrt{x})^2 + 1 = x + 1$$

$$(f \circ g)(x) = x + 1$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \sqrt{x^2 + 1}$$

$$(g \circ f)(x) = \sqrt{x^2 + 1}$$

$$f \circ g \neq g \circ f$$

Composition functions

- **Example 2:** $f(x) = 2x$, $g(x) = \sqrt{x-5}$, $h(x) = \frac{1}{x}$

Find $(f \circ f)(0.1)$ and $(f \circ g \circ h)(1)$

$$\begin{aligned} & \underbrace{f \circ f}_{f(f(0.1))}(0.1) \\ &= f(0.2) \\ &= 2 \times 0.2 \\ &= 0.4 \end{aligned}$$

$$\begin{aligned} h(1) &= \frac{1}{1} = 1 \\ g(1) &= \sqrt{1-5} = \sqrt{-4} \quad \times \\ (f \circ g \circ h)(1) & \text{ not defined!} \\ & 1 \text{ lies outside of the domain.} \end{aligned}$$

Composition functions

- **Example 2:** $f(x) = 2x$, $g(x) = \sqrt{x-5}$, $h(x) = \frac{1}{x}$

Find $f \circ g \circ h$ and its domain.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\frac{1}{x}\right)\right) = f\left(\sqrt{\frac{1}{x}-5}\right) \\ = 2\sqrt{\frac{1}{x}-5} = (f \circ g \circ h)(x)$$

Domain: $x \neq 0$, $\frac{1}{x}-5 \geq 0$

Composition functions

- **Example 2:** $f(x) = 2x$, $g(x) = \sqrt{x-5}$, $h(x) = \frac{1}{x}$

Find $f \circ g \circ h$ and its domain.

$$\frac{1}{x} - 5 = \frac{1-5x}{x}$$

Sign chart:

x	0	$\frac{1}{5}$	
$1-5x$	+	0	-
x	-	0	-

For $f \circ g \circ h$ to be well-defined

$$\begin{cases} x \neq 0 \\ \frac{1}{x} - 5 \geq 0 \end{cases}$$

$$\begin{cases} x \neq 0 \\ 0 \leq x \leq \frac{1}{5} \end{cases}$$

$$0 < x \leq \frac{1}{5}$$

$$x \in (0, \frac{1}{5}]$$

Domain