# Functions involving radicals

# Radicals - review

- The n'th root: $\sqrt[n]{x}$ is a number which, when raised to the power n, is equal to x. When there exist two of such numbers, pick the positive one.
- Examples:

$(-3)^2 = (-3)(-3) = 9$

$$\sqrt[2]{9} = \sqrt{9} = 3$$

$2^3 = 8$

$$\sqrt[3]{8} = 2, \qquad \sqrt[3]{-8} = -2$$

$(-2)^3 = -8$

$$\sqrt[4]{-2} \text{ is undefined}$$

$x^4 \neq -2$

# Radical functions - domain

- The ***domain*** of the function $f(x) = \sqrt[n]{x}$ is

$[0, \infty)$ if n is even (2, 4, 6, 8,….)

$(-\infty, \infty)$ if n is odd (3, 5, 7, 9,…)

- The radical function $\sqrt[n]{x}$ is the **inverse** of the power function $x^n$

$\sqrt{9} = 3$ $3^2 = 9$

$(-3)^2 = 9$

$x \longrightarrow \boxed{\sqrt[n]{\ }} \longrightarrow y$

$\sqrt[2]{x} = \sqrt{x}$ , $x \geq 0$

$\sqrt[3]{-1} = -1$ because $(-1)^3 = -1$

$\sqrt[3]{1} = 1$ $1^3 = 1$

index radical radicand

$-27 \longrightarrow \boxed{\sqrt[3]{\ }} \longrightarrow -3$

$\boxed{{}^{\wedge}3}$

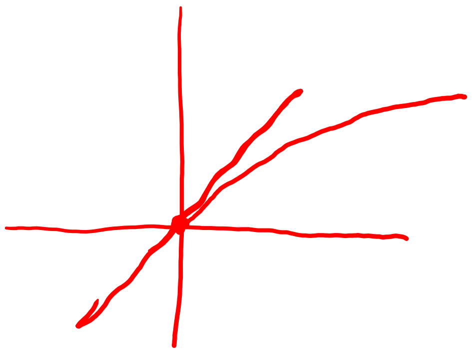
$x \longrightarrow \boxed{\sqrt[n]{\ }} \longrightarrow y \longrightarrow \boxed{?} \longrightarrow x$

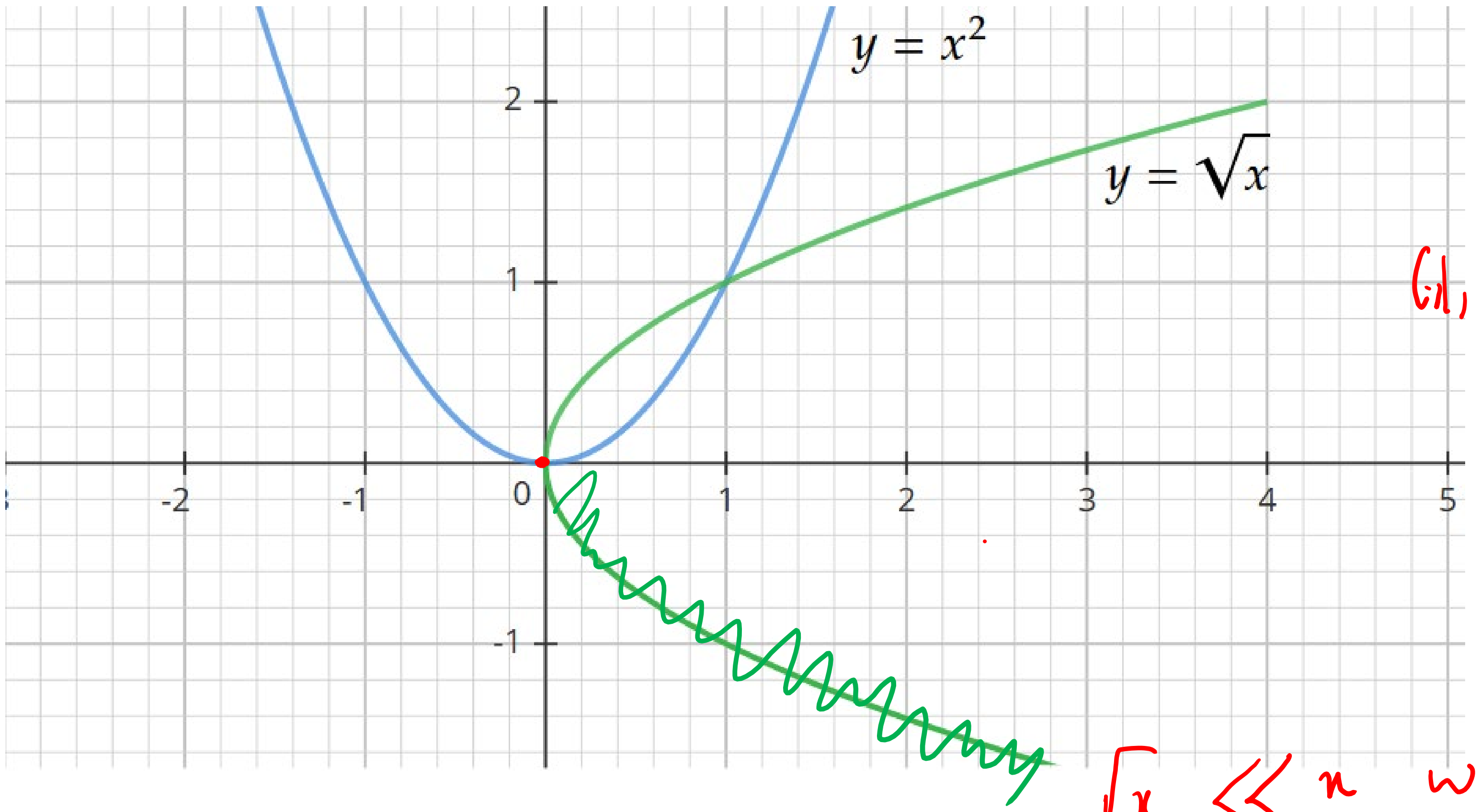
$9 \longrightarrow \boxed{\sqrt{\ }} \longrightarrow 3 \longrightarrow \boxed{{}^{\wedge}2} \longrightarrow 9$

# Radical functions – graph

$f(x) = \sqrt{x}$

$x^2$






$(1, 1)$ on $y = \sqrt{x}$

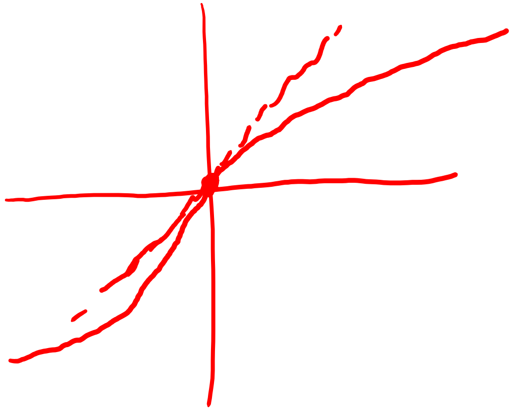
$1 = \sqrt{1}$

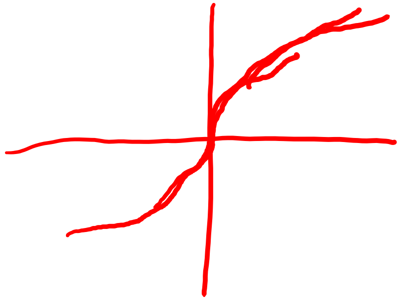
$\sqrt{x} \ll x$ when $x$ large

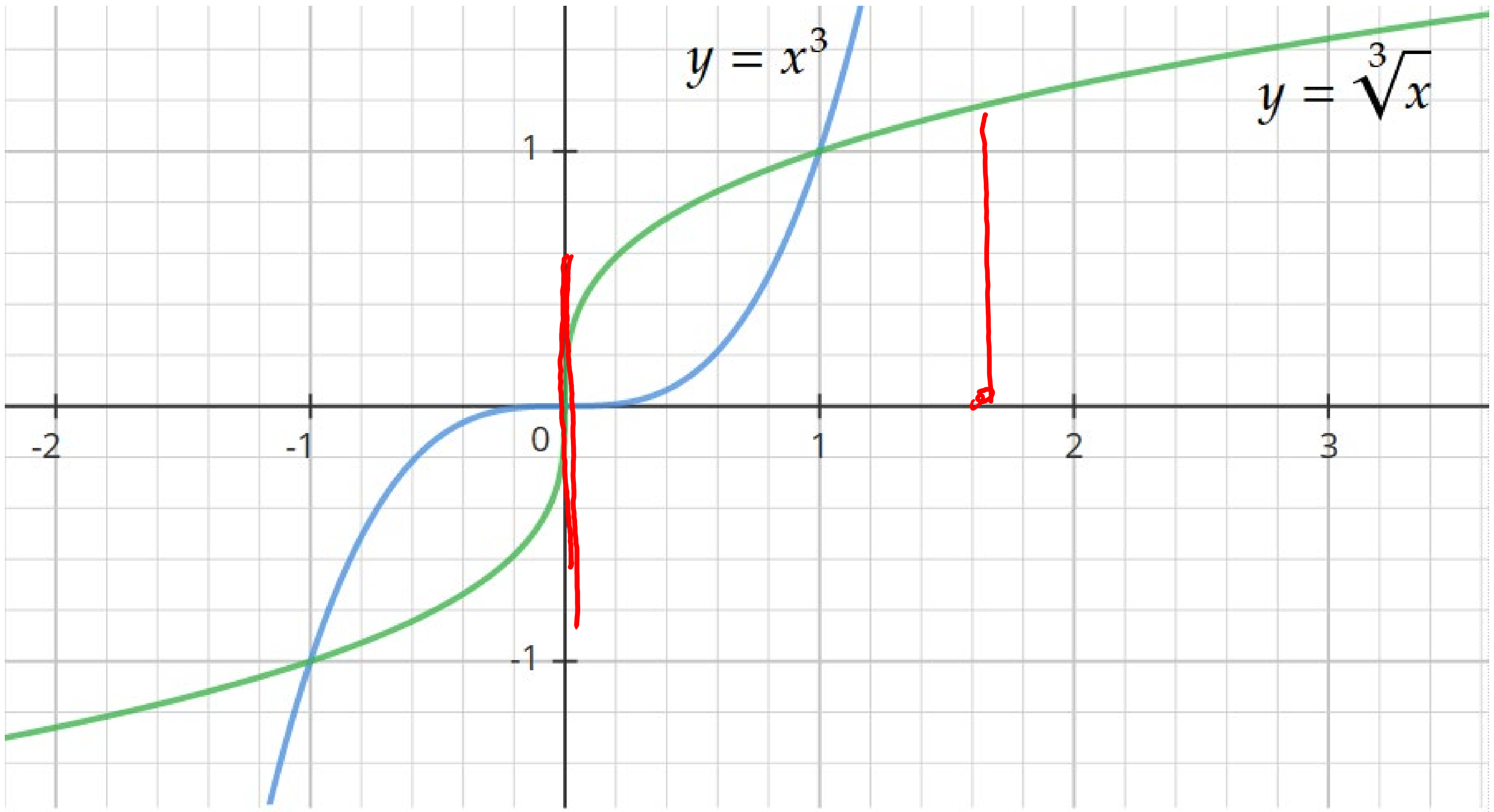
# Radical functions – graph

$y = x$

$\sqrt[3]{x} \longleftrightarrow x^3$

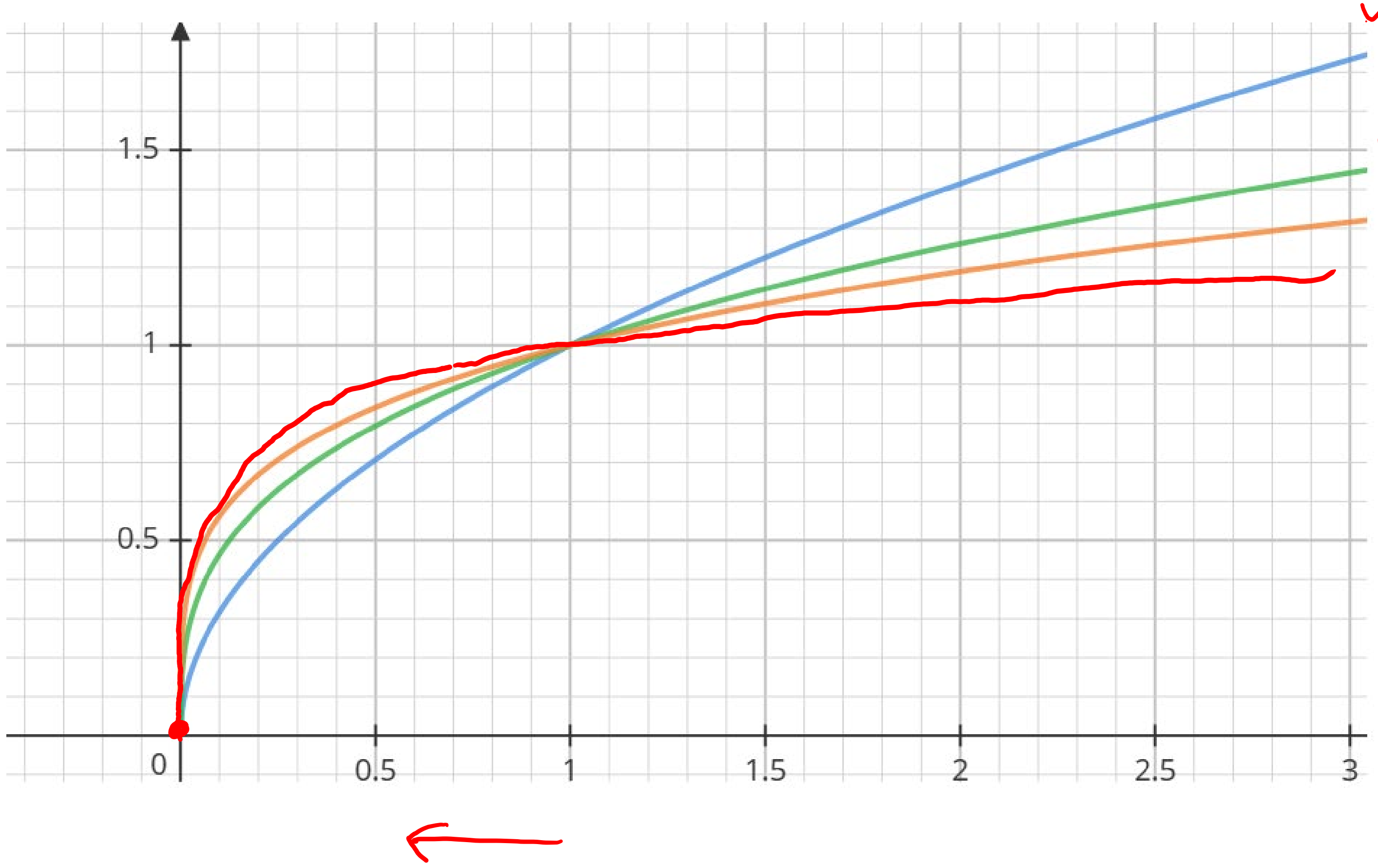






# Radical functions – graph

$f(x) = \sqrt[10]{x}$




$y = \sqrt{x}$

$y = \sqrt[3]{x}$

$y = \sqrt[5]{x}$

$y = \sqrt[n]{x}$

# Radical functions – power form

$$x^{m/n} = \sqrt[n]{x^m}$$

$x^{\boxed{m}/n} = \sqrt[n]{x^m}$

$\left(x^{1/2}\right)^2 = x^{\frac{1}{2}} x^{\frac{1}{2}}$

$= x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$

$= x^{-n+n} = x^0 = 1$

Examples:

$$x^{1/2} = \sqrt{x}$$ Domain $= [0, \infty)$

$$x^{1/3} = \sqrt[3]{x}$$ Domain $= (-\infty, \infty)$

$$x^{-1/3} = \frac{1}{x^{1/3}} = \frac{1}{\sqrt[3]{x}}$$ Domain $= (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}.$

$$x^{-2/3} = \frac{1}{x^{2/3}} = \frac{1}{\sqrt[3]{x^2}}$$ Domain $= (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}$

# Radical functions – power form

$$x^{m/n} = \sqrt[n]{x^m}$$

$- \quad = \frac{1}{(-8)^2} = \frac{1}{8)^2}$

Examples:

$x^3 \geq 0$

$$x^{3/4} = \sqrt[4]{x^3}$$ Domain : $x \geq 0$ $\quad [0, \infty)$

$$x^{-3/4} = \frac{1}{x^{3/4}} = \frac{1}{\sqrt[4]{x^3}}$$ Domain:

need $\begin{cases} \sqrt[4]{x^3} \neq 0 \\ x^3 \geq 0 \end{cases} \longrightarrow \begin{cases} x \neq 0 \\ x \geq 0 \end{cases}$

$x > 0$

Domain = $(0, \infty)$.

# Radical functions – Review some identities

$$\boxed{\sqrt{x^2} = |x|}$$

$\sqrt{3^2} = \sqrt{9} = 3$

$\sqrt{(-3)^2} = \sqrt{9} = 3$

$\sqrt{x^2}$ not always $= x$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$|-3| = 3$

$|3| = 3$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \qquad \begin{pmatrix} a \geq 0 \\ b > 0 \end{pmatrix}$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad a, b \geq 0$$

$\sqrt{\frac{9}{(-5)^2}} = \frac{\sqrt{9}}{\sqrt{(-5)^2}} = \frac{3}{\sqrt{(-5)(-5)}} = \frac{3}{\sqrt{25}} = \frac{3}{5}.$

# Radical functions – composition

- Example 1: find the domain of the function

$$f(x)=\sqrt{\frac{x}{x^2-3x+2}}$$

For $f$ to be well-defined, we need: $\begin{cases} x^2-3x+2 \neq 0 \\ \frac{x}{x^2-3x+2} \geq 0 \end{cases}$

$x^2-3x+2 = (x-1)(x-2)$

$\begin{cases} (x-1)(x-2) \neq 0 \longrightarrow x-1 \neq 0 \text{ and } x-2 \neq 0 \\ \frac{x}{(x-1)(x-2)} \geq 0 \end{cases}$

$x \neq 1, 2$

# Radical functions – composition

- Example 1: find the domain of the function

$$f(x)=\sqrt{\frac{x}{x^2-3x+2}}\qquad g(x)$$

want $g(x) \geq 0.$

$$g(x) = \frac{x}{(x-1)(x-2)}$$

| $x$ | $-$ | 0 | $+$ | 1 | $+$ | 2 | $+$ |
|---|---|---|---|---|---|---|---|
| $x-1$ | $-$ | | $-$ | | $+$ | | $+$ |
| $x-2$ | $-$ | | $-$ | | $-$ | | $+$ |
| $g(x)$ | $-$ | 0 | $+$ | | $-$ | | $+$ |

want $0 \leq x < 1$ and $x > 2$.

Conclusion: domain $= [0,1) \cup (2,\infty)$.

# Radical functions – composition

- Example 2: find the domain of the function

$$f(x) = \frac{x^{1/4}}{\sqrt[3]{x^2 - 3x + 2}}$$

For $f$ to be well-defined

$$\begin{cases} x^2 - 3x + 2 \neq 0 \to x \neq 1; 2 \\ x \geq 0 \longrightarrow x \geq 0 \end{cases}$$

0 1 2

Domain of $f$ = $[0, 1) \cup (1, 2) \cup (2, \infty)$.

# Radical functions – composition

- Example 3: find the inverse of the function

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

$$y = \frac{x}{\sqrt{x^2+1}} \rightarrow \boxed{x = \frac{y}{\sqrt{y^2+1}}}$$

Solve for y.

x & y have the same sign

Square:

$$x^2 = \left(\frac{y}{\sqrt{y^2+1}}\right)^2 = \frac{y^2}{\left(\sqrt{y^2+1}\right)^2} = \frac{y^2}{y^2+1}$$

$$x^2 = \frac{y^2}{y^2+1}$$

solve for y

mult. both sides by $y^2+1$.

$$x^2(y^2+1) = y^2$$

# Radical functions – composition

- Example 3: find the inverse of the function

$$f(x)=\frac{x}{\sqrt{x^2+1}}$$

$$x^2(1+y^2)=y^2$$

$$x^2+x^2y^2=y^2$$

$$x^2=y^2-x^2y^2=y^2(1-x^2)$$

$$y^2=\frac{x^2}{1-x^2}$$

<u>square root</u>:

$$\pm y=\sqrt{\frac{x^2}{1-x^2}}=\frac{\sqrt{x^2}}{\sqrt{1-x^2}}=\frac{|x|}{\sqrt{1-x^2}}=\frac{\pm x}{\sqrt{1-x^2}}$$

$$y=\frac{\pm x}{\sqrt{1-x^2}}$$

$$y=\frac{x}{\sqrt{1-x^2}}$$

$$f^{-1}(x)=\frac{x}{\sqrt{1-x^2}}$$