# Exponential and logarithm

# Raising to a power

- Natural exponent: $a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$

- Negative integer exponent: $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$

- Rational exponent: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

$2^3 = 2(2)(2)$

$2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{2(2)(2)} = \dfrac{1}{2}\,\dfrac{1}{2}\,\dfrac{1}{2}$

$2^{\frac{3}{5}} = \sqrt[5]{2^3}$

$2^{\sqrt{2}} = ??$

# Raising to a power

- Irrational exponent: $a^x$ where $x$ is an irrational number

$2^{\sqrt{2}}$

$\sqrt{2} = 1.4142\ldots$

$2^{\sqrt{2}} \approx 2^{1.4}$

$2^{1.4} = 2^{\frac{14}{10}} = 2^{\frac{7}{5}} = \sqrt[5]{2^7} = \sqrt[5]{128}$

$2^{\sqrt{2}} \approx 2^{1.41} = 2^{\frac{141}{100}} = \sqrt[100]{2^{141}}$

$2^{\sqrt{2}} \approx 2^{1.414}$

$2^{\sqrt{2}} \approx 2^{1.4142}$

$\ldots$

$2^x$

# Exponential function

given a number $a > 0$

$a^x$

$2^x$ $3^x$ $\left(\frac{1}{2}\right)^x$

- The function $a^x$ is called the *exponential function* with base $a$ .

base

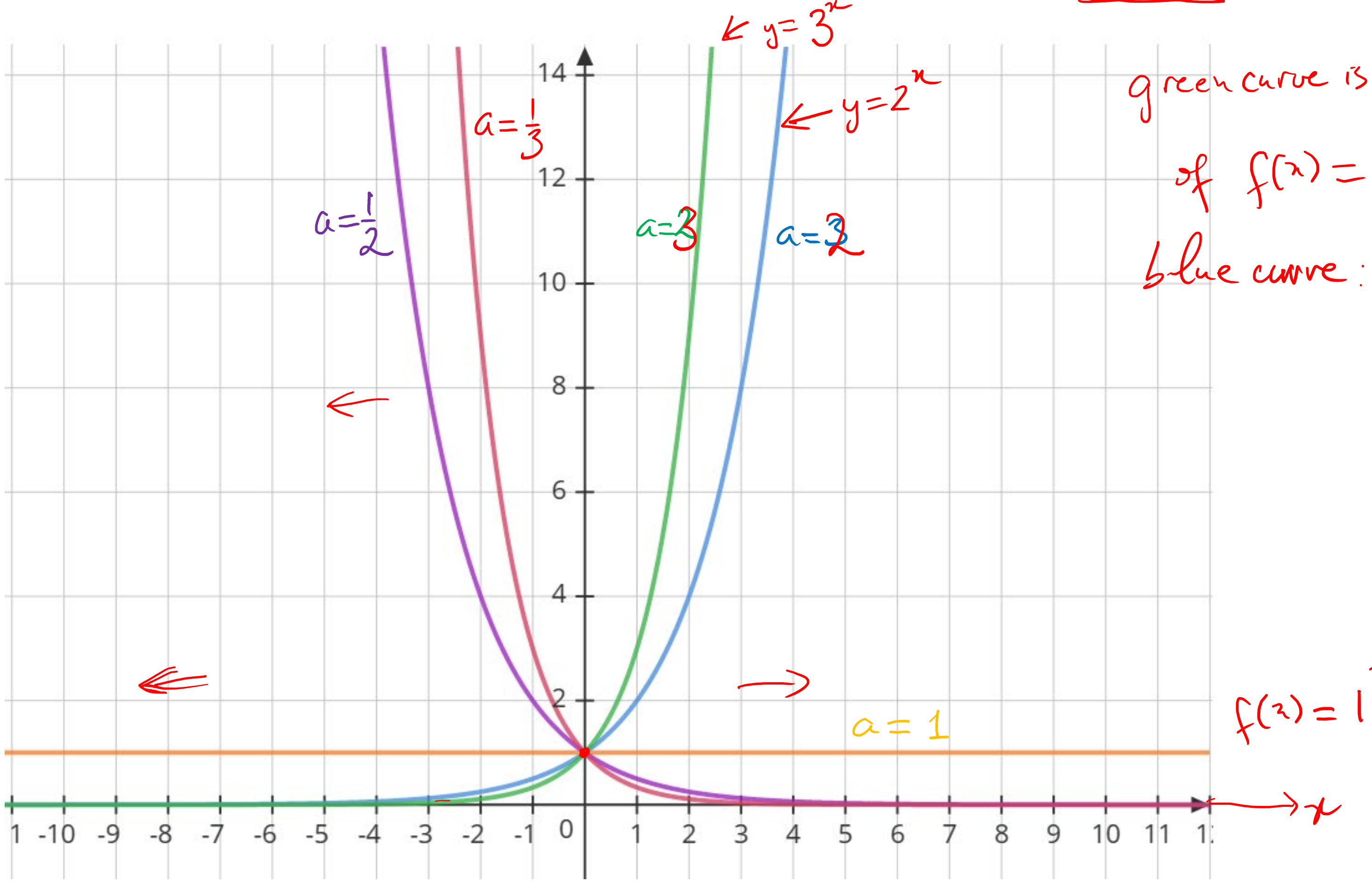
exponent

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$

$a = 1$ $a^x = 1^x = 1$

$a \neq 1$

$a^0 = 1$



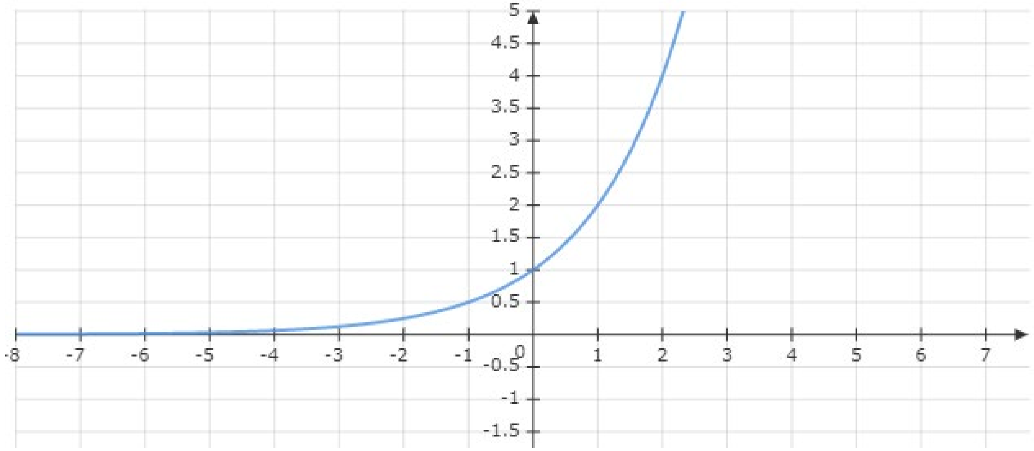

green curve is the graph of $f(x) = 3^x$.

blue curve: $f(x) = 2^x$

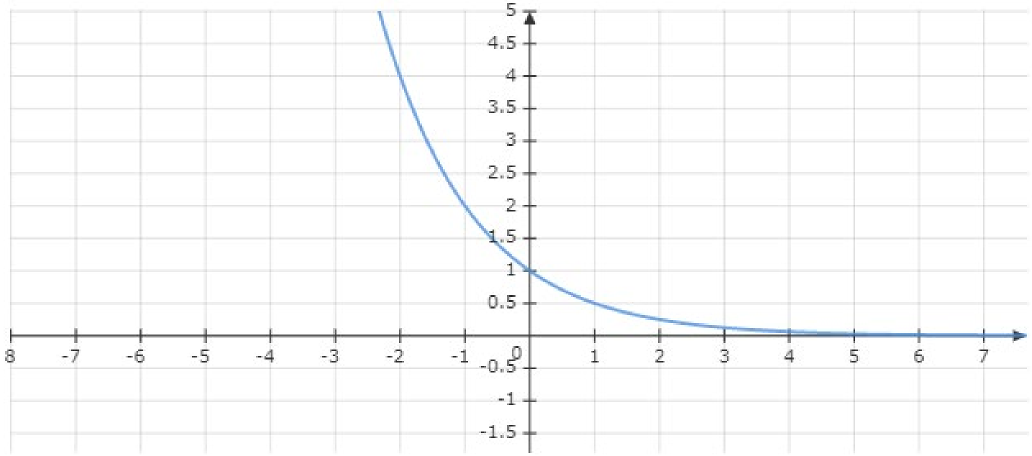
$f(x) = 1^x = 1$

# Exponential function

- If $a > 1$, the function $a^x$ is increasing:




- If $0 < a < 1$, the function $a^x$ is decreasing:




$(-1)^3 \quad (-1)^2 \quad (-1)^{\frac{1}{3}} = \sqrt[3]{-1} = -1$

$a < 0 \rightarrow$ not considered

$a = 0$

$(-1)^{\frac{1}{4}}$ not defined

$= \sqrt[4]{-1}$ ✗

# Algebraic properties of the exponential

$a^x$

- $1^c = 1$
- $0^c = 0 \quad \text{if} \quad c > 0$

$0^0$ undefined

$0^{-1} = \frac{1}{0}$ undefined

- $a^{-b} = \frac{1}{a^b}$
- $(ab)^c = a^c b^c, \qquad \left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$
- $\boxed{(a^b)^c = a^{bc}}$

# Algebraic properties of the exponential

- **Example:** simplify

$$\left(\frac{8x^3}{27}\right)^{-\frac{1}{3}} = \frac{(8x^3)^{-1/3}}{27^{-1/3}} = \frac{8^{-1/3}(x^3)^{-1/3}}{27^{-1/3}}$$

$$= \frac{8^{-1/3}\; x^{3(-\frac{1}{3})}}{27^{-1/3}} = \frac{8^{-1/3}\; x^{-1}}{27^{-1/3}} = \frac{\frac{1}{2}\,\frac{1}{x}}{\frac{1}{3}} = \frac{1}{2}\,\frac{1}{x}\,3 = \boxed{\frac{3}{2x}}$$

$$8 = 2^3 \longrightarrow 8^{-1/3} = (2^3)^{-1/3} = 2^{3(-1/3)} = 2^{-1} = \frac{1}{2}$$

$$27 = 3^3 \longrightarrow 27^{-1/3} = (3^3)^{-1/3} = 3^{3(-1/3)} = 3^{-1} = \frac{1}{3}$$

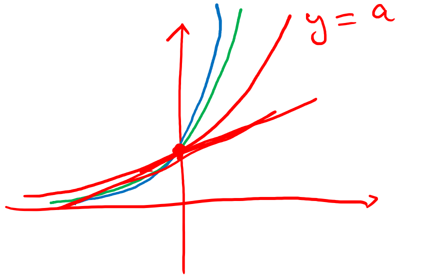
# Two special bases

- Common base: $a = 10$

$10^2$, $10^3$, $10^6$, $10^9$

$10^{-6}$, $10^{-2}$, $10^{-9}$

- Natural base: $a = e \approx \boxed{2.71828}\ldots$




Euler

Nepier

$(e^x)' = e^x$

# Logarithm function

$2^3 = 8$ $a=2$, $b=3$, $c=8$

If $a^b = c$ then we write $\mathrm{b} = \log_a \mathrm{c}$

- Examples:

$\log_3 1 =$

$\log_2 8 =$

$\log_{\frac{1}{4}} 32 =$

Ex

$$3^{\log_3 7} = ?$$

$$a^b = c \longleftrightarrow b = \log_a c$$

$$a^{\overbrace{\boxed{\log_a c}}^{b}} = c$$

$$\boxed{a^{\log_a c} = c}$$

$$\underbrace{3}_{=a}^{\log_{\underset{=a}{3}} \overset{=c}{7}} = 7$$

# Logarithm function - graph

- Graph of the function $\log_3 x$

| $x$ | $log_3 x$ |
|---|---|
| 1 | 0 |
| 3 | 1 |
| 9 | 2 |
| 1/3 | -1 |
| 1/9 | -2 |
| 1/27 | -3 |
| 1/81 | -4 |

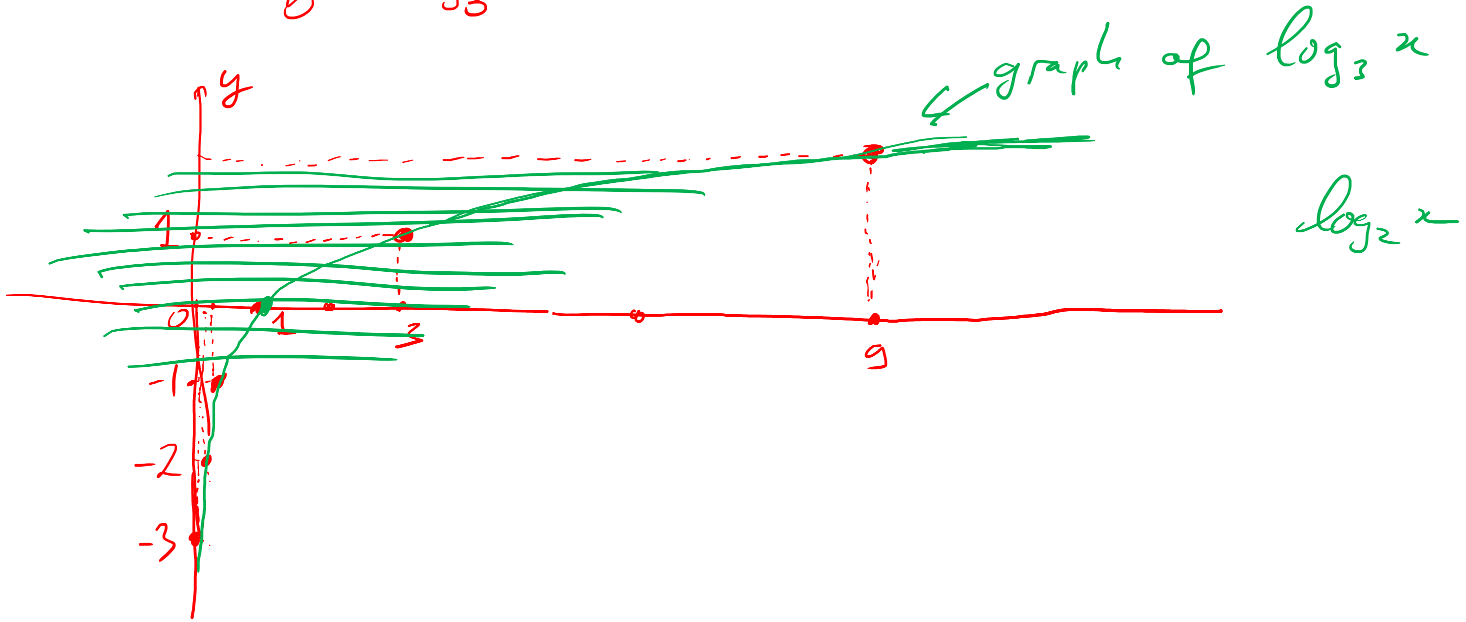
$3 = 3^{1}$ ($c = 3$, $a = 3$, $b = 1$)

$\log_3 3 = 1$

$\log_3 3 =$

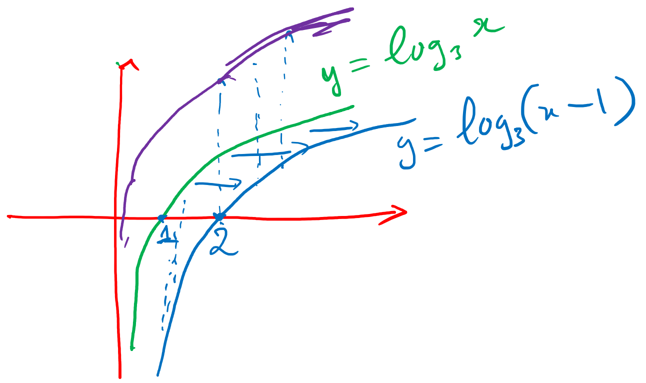
$3^{-1} = \frac{1}{3}$ ($a = 3$, $b = -1$, $c = \frac{1}{3}$)

$b = \log_3 \frac{1}{3}$



# Logarithm function - graph

**Example:** Sketch the graph of the function $f(x) = \log_3(x-1) + 2$



# Logarithm function – domain and range

The function $\log_a x$ has

$a^x$

- domain : $(-\infty, \infty)$
- range : $(0, \infty)$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$

# Logarithm function – domain and range

$\underbrace{\ln}_{\text{natural logarithm}} x = \log_e x \qquad e \approx 2.71828\ldots$

$\ln 0$ is undefined

**Example:** Find the domain of the function

$$f(x) = 3^{x+5} + \log_2(\underbrace{x^2 - 1}_{>0}) - 2\ln\left(\frac{x}{\underbrace{x-2}_{>0}}\right)$$

Need $x^2 - 1 > 0$, $\frac{x}{x-2} > 0$

Need $(x-1)(x+1) > 0$, $\frac{x}{x-2} > 0$

| $x$ | | $-1$ | | $0$ | | $1$ | | $2$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x-1$ | $-$ | | $-$ | | $-$ | $0$ | $+$ | | $+$ |
| $x+1$ | $-$ | $0$ | $+$ | | $+$ | | $+$ | | $+$ |
| $x$ | $-$ | | $-$ | $0$ | $+$ | | $+$ | | $+$ |
| $x-2$ | $-$ | | $-$ | | $-$ | | $-$ | $0$ | $+$ |

Conclusion:

$x < -1$ or $x > 2$

Domain $= (-\infty, -1) \cup (2, \infty)$.