# Properties of logarithm

# Exponential function - review

base  exponent

- $a^b a^c = a^{b+c}$, $\underbrace{\frac{a^b}{a^c} = a^{b-c}}_{b=c}$
- $(a^b)^c = a^{bc}$
- $a^0 = 1$
- $a^1 = a$
- $a^{m/n} = \sqrt[n]{a^m}$

$\frac{a^b}{a^b} = a^{b-b}$

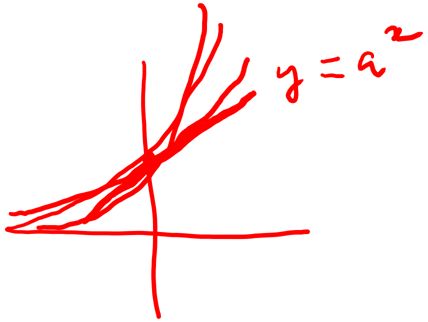
$\underbrace{\frac{a^b}{a^b}}_{1} = a^0$

$a^{2/3} = \sqrt[3]{a^2}$

$m=1,\ n=2$

# Logarithm - review

- $a^b = c$ if and only if $b = \log_a c$
- $\log_a 1 = 0$ $\Longleftrightarrow$ $a^0 = 1$
- $a^{\log_a c} = c$
- $\boxed{\log_a\left(a^b\right) = b}$ $a = 3,\ b = 0$




- $\log x = \log_{10}(x)$
  common base
- $\ln x = \log_e(x)$, where $e \approx 2.71828\ldots$

# Logarithm - review

$81 = 9 \times 9 = 3^2 \times 3^2 = 3^{2+2} = 3^4$

$\sqrt{3} = \sqrt[2]{3^1} = 3^{\frac{1}{2}}$

$\frac{1}{\sqrt{3}} = \frac{1}{3^{1/2}} = \frac{3^0}{3^{1/2}} = 3^{0-\frac{1}{2}} = 3^{-\frac{1}{2}}$

- $\log_3 1 =$ $\log_3 3^0 = 0$
- $\log_3 81 =$ $\log_3(3^4) = 4$
- $\log_3(\sqrt{3}) =$ $\log_3(3^{\frac{1}{2}}) = \frac{1}{2}$
- $\log_3\left(\frac{1}{\sqrt{3}}\right) =$ $\log_3(3^{-1/2}) = -\frac{1}{2}$

# Logarithm – algebraic properties

1) $\log_a(xy) = \log_a x + \log_a y$

2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3) $\log_a(x^r) = r\log_a x$

$\underbrace{a^b}_{x}\,\underbrace{a^c}_{y} = \underbrace{a^{b+c}}_{xy}$

$a^b = x \rightarrow b = \log_a x$

$a^c = y \rightarrow c = \log_a y$

$b+c = \log_a x + \log_a y$

$a^{b+c} = xy \rightarrow b+c = \log_a(xy)$

# Logarithm – algebraic properties

Conclusion :

$$\boxed{12 - 12\log_6 x - 4\log_6 y}$$

- **Example 1:** Expand the following logarithm

$$\log_6\left(\frac{216}{x^3y}\right)^4 = 4\log_6\left(\frac{216}{x^3y}\right) = 4\left(\overbrace{\log_6 216}^{3} - \log_6(x^3y)\right)$$

$$216 = 36\times 6 = 6^2\times 6^1 = 6^{2+1} = 6^3.$$

$$\log_6 216 = \log_6(6^3) = 3$$

$$= 4\left(3 - \log_6(x^3y)\right) = 4\left(3 - \left(\log_6 x^3 + \log_6 y\right)\right)$$

$$= 4\left(3 - \log_6 x^3 - \log_6 y\right) = 4\left(3 - 3\log_6 x - \log_6 y\right)$$

# Logarithm – algebraic properties

- **Example 2:** Expand the following logarithm

$$\ln\left(\frac{\sqrt[3]{x}}{10\sqrt{yz}}\right) = \underbrace{\ln\sqrt[3]{x}}_{\ln(x^{\frac{1}{3}})} - \underbrace{\ln(10\sqrt{yz})}_{\ln 10 + \ln\sqrt{yz}}$$

$$= \frac{1}{3}\ln x$$

$$\ln 10 + \ln\sqrt{yz} = \ln 10 + \ln\left((yz)^{\frac{1}{2}}\right)$$

$$= \ln 10 + \frac{1}{2}\ln(yz)$$

$$= \ln 10 + \frac{1}{2}(\ln y + \ln z)$$

$$\frac{1}{3}\ln x - \left(\ln 10 + \frac{1}{2}(\ln y + \ln z)\right) = \boxed{\frac{1}{3}\ln x - \ln 10 - \frac{1}{2}(\ln y + \ln z)}$$

# Logarithm – algebraic properties

- **Example 3:** Write the following expression as a single logarithm

$$\underbrace{3\log_2(x)}_{\log_2(x^3)} - \underbrace{\frac{1}{2}\log_2(y)}_{\log_2(y^{\frac{1}{2}})} + \frac{1}{2}$$

$c = \log_a(a^c)$

$c = \frac{1}{2}$, $a = 2$

$\frac{1}{2} = \log_2\left(2^{\frac{1}{2}}\right)$

$= \log_2(\sqrt{2})$

$= \log_2(x^3) - \log_2(\sqrt{y}) + \log_2\sqrt{2}$

$= \log_2 \frac{x^3}{\sqrt{y}} + \log_2\sqrt{2} = \log_2\left(\frac{x^3}{\sqrt{y}}\sqrt{2}\right)$ ✓

# Logarithm – algebraic properties

- **Example 4:** Simplify

$\underbrace{e}_{a}^{\overbrace{2}^{c}\,\overbrace{\ln x}^{b}} = \left(e^{\ln x}\right)^2$

$= \boxed{x^2}$

$(a^b)^c = a^{bc}$

$e^{\ln x} = e^{\log_e x} = x$

# Change of bases

$a = c^{\log_c a}$

$$\underbrace{a^b}_{x} = \left(c^{\log_c a}\right)^b = \underbrace{c^{b\log_c a}}_{x}$$

$a^b = x \longrightarrow b = \log_a x$

$c^{\boxed{b\log_c a}} = x \longrightarrow b\log_c a = \log_c x$

$\longrightarrow b = \dfrac{\log_c x}{\log_c a} \longrightarrow \log_a x = \dfrac{\log_c x}{\log_c a}$

$\log_a x = \dfrac{\log_c x}{\log_c a}$

# Change of bases

$$\boxed{a^b = c^{b\log_c a}}$$

$$\boxed{\log_a c = \frac{\log_b c}{\log_b a}}$$

$\log_2 3 = ?$

$b = e$: $\log_a c = \dfrac{\ln c}{\ln a}$

**Examples:**

$\log_2(3) = \dfrac{\log_e(3)}{\log_e(2)} = \dfrac{\ln 3}{\ln 2} \approx 1.585$

$\log_{\frac{1}{100}}(2) = \dfrac{\log 2}{\log \frac{1}{100}} = \dfrac{\log 2}{\log\left(\frac{1}{10^2}\right)} = \dfrac{\log 2}{\log(10^{-2})} = \dfrac{\log 2}{-2} \approx -0.1505\ldots$

# Change of bases

$$\boxed{\underset{5}{a}^{\overset{x-1}{b}} = \underset{10}{c}^{b\log_c a}}$$

$$\boxed{\log_a c = \frac{\log_b c}{\log_b a}}$$

**Examples:**

$$\underbrace{5^{x-1}} = e^{(x-1)\log_e 5} = \underbrace{e^{(x-1)\ln 5}}$$

$$\underbrace{5^{x-1}} = 10^{(x-1)\log_{10} 5} = \underbrace{10^{(x-1)\log 5}}$$