Lecture 22

Monday, February 20, 2023 7:59 AM

* Questions

L'Hopital rule

This is a rule that helps us find limits in an indeterminate form $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{\infty}{-\infty}$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if this limit exists or equal to ∞ or $-\infty$.

Explanation: we will need Cauchy's thm (a generalization of Mean Value theorem) to prove this rule.

However, the case $g'(a) \neq 0$, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ has a simple explanation:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

Ex: $\lim_{x\to 0} \frac{\sin x}{x}$ is of the form $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = 1$$

Ex: $\lim_{x\to\infty} \frac{\sin x}{x}$ is not of an indeterminate form, so we can't use L'Hospital rule.

Ex $\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$ is of the form $\frac{0}{0}$

$$= \lim_{x\to -3} \frac{2x+2}{2x+4} = \frac{2(-3)+2}{2(-3)+4} = \frac{5}{2}$$

Ex $\lim_{x\to 1} \frac{x^3-3x^2+3x-1}{x^2-2x+1}$ is of the form $\frac{0}{0}$

$$= \lim_{x\to 1} \frac{3x^2-6x+3}{2x-2}$$ (also of the form $\frac{0}{0}$)

$$= \lim_{x\to 1} \frac{6x-6}{2} = \frac{6(0)-6}{2} = 0$$

Ex $\lim_{x\to 0^+} \frac{\ln x}{x}$ is not of an indeterminate form, so we can't use L'Hospital rule.

Ex $\lim_{x\to\infty} \frac{\ln x}{x}$ is of the form $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Observation As $x\to\infty$, both x and $\ln x$ go to infinity. However, $\ln x$ goes to infinity much slower than x.

Ex $\lim_{x\to\infty} xe^{-x}$ is of the form 0∞, which is also indeterminate.

But there is no fraction, so we can't use L'Hospital rule (yet!)

Notice that $e^{-x} = \frac{1}{e^x}$, so

$$xe^{-x} = \frac{x}{e^x}$$

Now we can use L'Hospital rule.

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Observation: e^x grows much faster than x as $x\to\infty$.

Ex

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

$$\frac{\infty}{\infty}$$

Observation: e^x grows much faster than x^2 as $x\to\infty$. In fact, e^x grows much faster than any polynomial as $x\to\infty$.