Lab 2

In this lab, we will practice with Mathematica the following topics:

- Visualize 2D/3D curves.
- Draw the domain of a function and level sets.
- Find limits of a function.
- Find partial derivatives.
- Visualize tangent planes.

1 Reminder about getting access

There are two ways to get free access to Mathematica:

A) Install three free components: *Wolfram Engine*, *JupyterLab*, and *WolframLanguageForJupyter*. You can use the unlimited computing power of Mathematica on your own computer, with Jupyter Notebook acting as a user interface. The instruction is here:

   https://web.engr.oregonstate.edu/~phamt3/Resource/Wolfram-Mathematica-with-JupyterLab.pdf

B) Use the cloud-based version of Mathematica: https://www.wolframcloud.com
   In this option, you are limited to about 8 minutes of computation per month. Files stored on the cloud will be deleted after 60 days.

2 For students using Jupyter Notebook

On Jupyter Notebook, in order to use the interactive features of Mathematica, such as dragging or rotating 3D graphs with your mouse, do the simple one-time procedure below:

- First, make sure that you are signed in on Wolfram Cloud by going to

   https://www.wolframcloud.com

   If you are not signed in yet, then sign in with your Wolfram ID (email address) and password. Check the box "Remember me" so that you won't have to do it again in the future.

- Second, make sure that your Jupyter Notebook is connected to the same Wolfram account you are signed in with on the web browser. This is done by executing the following command in Jupyter Notebook:

   CloudConnect["your-WolframID", "your-password"]

   From now on, you can use any command that has interactive features in Mathematica by adding the prefix Interact@ to that command. For example, the command

   ```
   Interact@Plot3D[Sin[x+y], {x, -Pi, Pi}, {y, -Pi, Pi}]
   ```

   will let you drag and rotate the graph. Without the prefix Interact@, the graph is still shown on Jupyter Notebook but just not interactable.

3 Visualize 2D/3D curves

Given a parametric equation of a curve, we use the command **ParametricPlot** if it is a 2D curve, and the command **ParametricPlot3D** if it is a 3D curve.

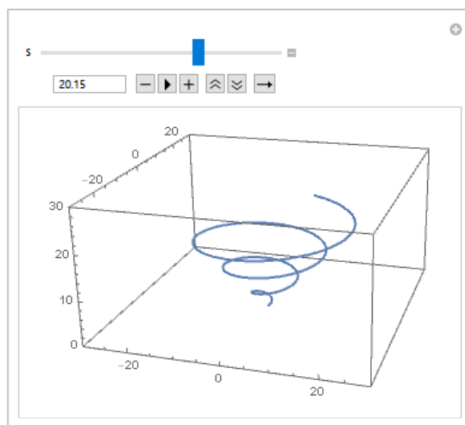(1) For example, to plot the curve $r(t) = (t\cos t,\ t\sin t,\ t)$, try the following:

```
ParametricPlot3D[{t*Cos[t], t*Sin[t], t}, {t, 0, 30}]
```
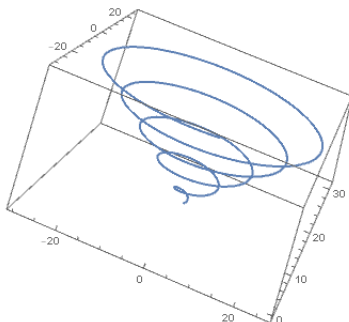

If you use Jupyter Notebook, add the prefix Interact@ to the command so that you can rotate the curve.

(2) The parametric equations also show us how a curve is drawn. For example, try the following:

```
Manipulate[ParametricPlot3D[{t*Cos[t], t*Sin[t], t}, {t, 0, s},
  PlotRange -> {{-30, 30}, {-30, 30}, {0, 30}}], {s, 0.1, 30}]
```


If you use Jupyter Notebook, add the prefix Interact@ to the command to be able to interact with the curve.

(3) Visualize the curve $r(t) = (\cos t,\ \sin t,\ \cos(2t))$ for $0 \le t \le 2\pi$.

(4) Visualize the curve $r(t) = (-2\cos t + \frac{4}{3}\cos(3t),\ 2\sin t + \frac{4}{3}\sin(3t))$ for $0 \le t \le 2\pi$. For what approximate values of t does the curve intersect itself? (This curve is on the logo of the Proximus company.)

4 Draw the domain and level sets of a function

(5) The command **RegionPlot** is used to draw a 2D region determined by one or more inequalities. For example, consider the function $f(x,y) = \sqrt{x^2 + y + 1}\sqrt{x - y - 1}$. The domain of this function is the set of points (x, y) such that $x^2 + y + 1 \geq 0$ and $x - y - 1 \geq 0$. Try the following:

```
RegionPlot[x^2 + y + 1 >= 0 && x - y - 1 >= 0, {x, -5, 5}, {y, -5, 5}]
```

(6) The command **RegionPlot3D** is used to plot a 3D region determined by one or more inequalities. For example, consider the function $g(x,y,z) = \sqrt{x + y + z}\sqrt{4 - x^2 - y^2 - z^2}$. The domain of this function is the set of points (x, y, z) such that $x + y + z \geq 0$ and $x^2 + y^2 + z^2 \leq 4$.

```
RegionPlot3D[x + y + z >= 0 && x^2 + y^2 + z^2 <= 4, {x, -2, 2},
            {y, -2, 2}, {z, -2, 2}, AxesLabel -> {x, y, z}]
```

(7) The command **ContourPlot** is used to draw the level sets of a function of two variables. The command **ContourPlot3D** is used to plot the level sets of a function of three variables. For example, consider the function $f(x,y)$ given above. Define the function:

```
f[x_, y_]  :=  Sqrt[x^2 + y + 1]*Sqrt[x - y - 1]
g[x_, y_, z_] := Sqrt[x + y + z]*Sqrt[4 - x^2 - y^2 - z^2]
```

(8) Draw one level set:

```
ContourPlot[ f[x, y] == 0.5, {x, -5, 5}, {y, -5, 5}]
ContourPlot[ g[x, y, z] == 1.5, {x, -5, 5}, {y, -5, 5}]
```

(9) Draw many level sets (the *contour map*): the lighter the color is where f has higher values.

```
ContourPlot[ f[x, y], {x, -5, 5}, {y, -5, 5}]
ContourPlot[ f[x, y], {x, -5, 5}, {y, -5, 5}, Contours->50]
ContourPlot[ f[x, y], {x, -5, 5}, {y, -5, 5}, Contours->{0.2, 1, 2, 3, 4}]
ContourPlot[ g[x, y, z], {x, -5, 5}, {y, -5, 5}]
```

(10) The surface $x^2 y + y^2 z + z^2 x = 1$ is the 1-level set of the function $f(x,y,z) = x^2 y + y^2 z + z^2 x$. Draw this surface.

5 Find limits of a function

(11) Type `Limit[(Cos[x]-1)/x^2, x->0]`, then `Shift+Enter`.

(12) Type `a = Limit[(1+1/x)^x, x->Infinity]`, then `Shift+Enter`.
Type `N[a,6]`, then `Shift+Enter`.
Type `N[a,10]`, then `Shift+Enter`.
Type `N[a,15]`, then `Shift+Enter`.

(13) Find the limit
$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$$
Note that in Mathematica, e^x is typeset as `E^x` or `Exp[x]`.

(14) Try `Limit[(x^3-y^3)/(x-y), {x,y}->{1,1}]`

3

(15) To find the limit along a chosen path, we use the *replacement operator* "/."

```
Limit[x^3/(x^2-y^2)/.{x->0, y->t},t->0]
Limit[x^3/(x^2-y^2)/.{x->t, y->2t},t->0]
Limit[x^3/(x^2-y^2)/.{x->t, y->t+t^2},t->0]
```

(x, y) goes to $(0, 0)$ along the y-axis in the first limit, along the line $y = 2x$ in the second limit, and along the parabola $y = x + x^2$ in the third limit.

(16) Explain why the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^2}{x - y}$$

does not exist. Graph the function $f(x, y) = \frac{y^2}{x-y}$ to see its behavior near the origin.

6 Find partial derivatives

(17) Type `D[Cos[x], x]`, then `Shift+Enter`.
Type `D[Cos[x], {x,2}]`, then `Shift+Enter`.
Type `D[Cos[x], {x,3}]`, then `Shift+Enter`.

(18) Find the third derivative of the function $f(x) = e^{\cos(x^2)}$.

(19) Try

```
f[x_, y_, z_]:= x^2 + y^3z + x*y*z^3
D[f[x,y,z], x, y]
D[f[x,y,z], {x,2}]
D[f[x,y,z], {x,1}, {z,2}]
```

7 Visualize tangent planes

(20) The elliptic paraboloid $z = x^2 + 4y^2$ contains the point $(1, -1, 5)$. The tangent plane to the surface at this point is $z = -5 + 2x - 8y$. We want to draw the surface and the plane on the same picture. Try the following:

```
p1 = Plot3D[x^2+4y^2, {x,-1,3}, {y,-3,1}]
p2 = Plot3D[-5+2x-8y, {x,-1,3}, {y,-3,1}, PlotStyle->Blue]
Show[p1, p2, BoxRatios->1]
```

If you use Jupyter Notebook, add the prefix Interact@ to the last command (Show) to be able to rotate the picture.

(21) Draw the surface $x^2y + y^2z + z^2x = 1$ together with the tangent plane to the surface at the point $(1, -1, 1)$.

8 To turn in

Submit your implementation of Exercises (1) - (21) as a single pdf file.