Lab 3

In this lab, we will practice with Mathematica the following topics:

- Visualize the gradient vector field and the level sets.
- Solve a system of equations.
- Visualize and classify critical points.
- Solve optimization problems.
- Evaluate double integrals.

1 Reminder about getting access

There are two ways to get free access to Mathematica:

A) Install three free components: *Wolfram Engine*, *JupyterLab*, and *WolframLanguageForJupyter*. You can use the unlimited computing power of Mathematica on your own computer, with Jupyter Notebook acting as a user interface. The instruction is here:

https://web.engr.oregonstate.edu/~phamt3/Resource/Wolfram-Mathematica-with-JupyterLab.pdf

B) Use the cloud-based version of Mathematica: https://www.wolframcloud.com
In this option, you are limited to about 8 minutes of computation per month. Files stored on the cloud will be deleted after 60 days.

2 For students using Jupyter Notebook

On Jupyter Notebook, in order to use the interactive features of Mathematica, such as dragging or rotating 3D graphs with your mouse, do the simple one-time procedure below. (**If you already did it in Lab 2, you don't need to do it again.**)

- First, make sure that you are signed in on Wolfram Cloud by going to

  https://www.wolframcloud.com

  If you are not signed in yet, then sign in with your Wolfram ID (email address) and password. Check the box "Remember me" so that you won't have to do it again in the future.

- Second, make sure that your Jupyter Notebook is connected to the same Wolfram account you are signed in with on the web browser. This is done by executing the following command in Jupyter Notebook:

  CloudConnect["your-WolframID", "your-password"]

  From now on, you can use any command that has interactive features in Mathematica by adding the prefix Interact@ to that command. For example, the command

  ```
  Interact@Plot3D[Sin[x+y], {x, -Pi, Pi}, {y, -Pi, Pi}]
  ```

  will let you drag and rotate the graph. Without the prefix Interact@, the graph is still shown on Jupyter Notebook but just not interactable.

3 Plot gradient vector field and level sets

Graphically speaking, a vector field is a map of arrows: at each point on the plane (or space) is placed a vector. Mathematically speaking, a vector field is a function of two or three variables taking vector values. For example, the gradient of the function $f(x,y) = x^2 + y^2$ is $\nabla f(x,y) = (2x, 2y)$, which is a vector field. At position (x,y) on the plane is placed the vector $(2x, 2y)$. To plot a vector field, use the command **VectorPlot**.

(1) Try the following:

```
VectorPlot[{2x, 2y}, {x, -2, 2}, {y, -2, 2}]
```

For visual effect, Mathematica scales all the arrows so that they are of the same length while preserving the direction of each arrow. The color reflects the actual length. The lighter the color, the longer the arrow.

(2) Plot the gradient vector field of the function $f(x,y) = \sin(xy)$ where $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$.

(3) Plot the contour map of the function f. If you forget how to do so, refer to Lab 2.

(4) Use command **Show** to show the gradient vector field and the contour map on the same plot. Do the gradient vectors look perpendicular to the level sets? Notice that they points to the direction where f increases the most.

4 Solve a system of equations

(5) The command **Solve** is used to solve an equation or a system of equations. Try the following:

```
Solve[ x^2-4x-6==0, x]
```

Note that we use the double equal sign to represent an equation. A single equal sign represents an assignment.

(6) Solve for the real root(s) of the polynomial $f(x) = x^3 + 2x - 5$.

(7) To solve a system of equations, we list all the equations of the system inside the command Solve, separated by &&. Try the following:

```
Solve[x+y==5 && x^2-y==3, {x,y}]
```

(8) To get numerical values instead of the exact solution, replace **Solve** by **NSolve**.

(9) Show the curve $x + y^2 = 1$ and the curve $x^2 - y^3 = 0$ on the same plot. Recall that to plot an equation, we use the command **ContourPlot**.

(10) Find all the intersection points between these two curves.

5 Optimization problem

(11) Let $f(x, y)$ be the function in Problem 5 of Section 11.7 (page 681). Plot this function in the region $-5 \leq x \leq 5$, $-5 \leq y \leq 5$. Refer to Lab 1 if you forget how to do so.

(12) Draw a contour map with 200 level sets. Refer to Lab 2 if you forget how to do so.

(13) How many critical points do you see? How many are local min/ local max/ saddle point?

(14) Find the partial derivatives of f.

(15) Use the command **Solve** as explained above to find all the critical points of f. Based on the contour map, which one(s) are local min/ local max/ saddle point?

(16) With the help of the contour map, determine the value(s) of (x, y) where the function attains minimum and maximum in the region $-5 \leq x \leq 5$, $-5 \leq y \leq 5$. What are the maximum and minimum values of f ?

6 Double integrals

The command **Integrate** is used to compute the exact integrals (definite or indefinite) in Mathematica. The command **NIntegrate** is used to compute approximate numerical value of a definite integral. For example,

(17) $\int \sin x dx$

```
Integrate[Sin[x], x]
```

(18) $\int_0^\pi \sin x dx$

```
Integrate[Sin[x], {x, 0, Pi}]
```

(19) $\int_0^2 \int_0^1 (x + 2y) dx dy$

```
Integrate[x + 2 y, {y, 0, 2}, {x, 0, 1}]
```

(20) $\int_0^2 \int_{y-1}^{2y} (x + 2y) dx dy$

```
Integrate[x + 2 y, {y, 0, 2}, {x, y - 1, 2 y}]
```

(21) Numerical approximation of $\int_0^2 \int_0^1 e^{-x^2 y^2} dx dy$

```
NIntegrate[Exp[-x^2 y^2], {x, 0, 1}, {y, 0, 2}]
```

(22) Evaluate the double integral in Problem 15 of Section 12.1 (page 705). Write the result as precise value as well as numerical value.

(23) Evaluate the double integral in Problem 8 of Section 12.2 (page 713). Write the result as precise value as well as numerical value.

7 To turn in

Submit your implementation of Exercises (1) - (23) as a single pdf file.