Lab 4

In this lab, we will practice with Mathematica the following topics:

- Sketching regions (in 2D) and solids (in 3D).
- Evaluating double and triple integrals.
- Sketching the new region/solid after a change of variables.
- Finding the Jacobian matrix and Jacobian determinant of a change of variables.

The term "solid" refers to a 3D object. The "region" can refer to either a 2D object or 3D object depending on the context.

1 Reminder about getting access

There are two ways to get free access to Mathematica:

A) Install three free components: *Wolfram Engine*, *JupyterLab*, and *WolframLanguageForJupyter*. You can use the unlimited computing power of Mathematica on your own computer, with Jupyter Notebook acting as a user interface. The instruction is here:

https://web.engr.oregonstate.edu/~phamt3/Resource/Wolfram-Mathematica-with-JupyterLab.pdf

B) Use the cloud-based version of Mathematica: https://www.wolframcloud.com
In this option, you are limited to about 8 minutes of computation per month. Files stored on the cloud will be deleted after 60 days.

2 For students using Jupyter Notebook

On Jupyter Notebook, in order to use the interactive features of Mathematica, such as dragging or rotating 3D graphs with your mouse, do the simple one-time procedure below. (**If you already did it in a previous lab, you don't need to do it again.**)

- First, make sure that you are signed in on Wolfram Cloud by going to

  https://www.wolframcloud.com

  If you are not signed in yet, then sign in with your Wolfram ID (email address) and password. Check the box "Remember me" so that you won't have to do it again in the future.

- Second, make sure that your Jupyter Notebook is connected to the same Wolfram account you are signed in with on the web browser. This is done by executing the following command in Jupyter Notebook:

  CloudConnect["your-WolframID", "your-password"]

  From now on, you can use any command that has interactive features in Mathematica by adding the prefix Interact@ to that command. For example, the command

  ```
  Interact@Plot3D[Sin[x+y], {x, -Pi, Pi}, {y, -Pi, Pi}]
  ```

  will let you drag and rotate the graph. Without the prefix Interact@, the graph is still shown on Jupyter Notebook but just not interactable.

3 Sketch regions and solids

(1) To declare a polygon (triangle, quadrilateral, pentagon, hexagon,...), we use the syntax

```
Polygon[{p1, p2, p3,...}]
```

where `p1,p2,p3,...` are the vertices of the polygon in that order. The command `Region` can be used to plot the region. Try the following:

```
R=Polygon[{{1,1},{2,1},{3,2},{-1,3}}]
Region[R, Axes->True, AxesOrigin->{0,0}]
```

The parts `Axes->True` and `AxesOrigin->{0,0}` in the above command are optional. Try `Axes->False` and/or `AxesOrigin->{1,1}` and see what happens.

(2) Plot the triangle with vertices at $(-1,1)$, $(2,0)$, $(1,2)$.

(3) You can also plot a polygon in 3D. Try the following:

```
S=Polygon[{{1,0,3},{0,2,3},{1,2,0}}]
Region[S, Axes->True,
          AxesOrigin->{0,0,0},
          AxesStyle->Thick,
          AxesLabel->{x,y,z}]
```

The parts `Axes->True`, `AxesOrigin->{0,0,0}`,... in the above command are optional.

(4) Regions and solids are usually described in terms of inequalities. For example, the ellipse centered at the origin, meeting the x-axis at $x = \pm 4$ and the y-axis at $y = \pm 3$ is described as

$$\frac{x^2}{16} + \frac{y^2}{9} \leq 1.$$

You can define this ellipse using `ImplicitRegion` and plot it using `Region` as follows:

```
R = ImplicitRegion[x^2/16+y^2/9<=1, {x,y}]
Region[R, Axes->True, AspectRatio->Automatic, AxesLabel->{x,y}]
```

(5) The part the the ellipse in the first quadrant is described by the inequalities

$$\frac{x^2}{16} + \frac{y^2}{9} \leq 1, \ x \geq 0, \ y \geq 0.$$

These inequalities are separated by `&&` as follows.

```
R = ImplicitRegion[x^2/16+y^2/9<=1 && x>=0 && y>=0, {x,y}]
Region[R, Axes->True, AspectRatio->Automatic, AxesLabel->{x,y}]
```

(6) Define and plot the upper half of the unit disk centered at the origin.

(7) Define and plot the 2D region above the parabola $y = x^2$ and below the line $y = 4$.

(8) Plotting a solid is a bit trickier, not because it can't be done the same way as for regions, but because we want good visual effects. Let R be the solid between two surfaces $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. This solid can be defined as

```
R = ImplicitRegion[x^2+y^2<=z<=8-x^2-y^2, {x,y,z}]
```

You then use the command `RegionPlot3D` to plot this solid:

```
RegionPlot3D[R, Axes->True,
                Mesh -> False,
                AxesOrigin->{0,0,0},
                AxesLabel->{x,y,z},
                PlotStyle->Opacity[0.5],
                AxesStyle->Thick,
                Boxed->False]
```

Adjust some parameters in the command (for example, `Boxed->True`, `Opacity[0.1]`,...) to see what happens. If you use Jupyter Notebook, add the prefix `Interact@` in front of `RegionPlot3D` to be able to rotate the solid.

(9) Define and plot the region inside the ellipsoid $x^2 + 4y^2 + z^2 = 9$.

(10) Define and plot the region inside the ellipsoid $x^2+4y^2+z^2 = 9$ in the first octant (i.e. $x, y, z \geq 0$).

(11) You can notice that the solids in the previous examples are not very smooth. Sometimes, simplifying the inequalities before defining the solid will help.

For example, consider the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x - y + 2z = 4$, in the first octant. We simplify the inequalities using `Reduce`.

```
ineq = Reduce[x^2 + y^2 <= 4 && x - y + 2 z <= 4 &&
              x >= 0 && y >= 0 && z >= 0, {x, y, z}]
```

Then define the solid as

```
R = ImplicitRegion[ineq,{x,y,z}]
```

Then plot the solid

```
RegionPlot3D[ineq, {x,0,2}, {y,0,2}, {z,0,4},
        PlotPoints -> 100,
        Axes->True,
        Mesh -> False,
        AxesOrigin->{0,0,0},
        AxesLabel->{x,y,z},
        PlotStyle->Opacity[0.5],
        AxesStyle->Thick,
        Boxed->False]
```

(12) Plot the solid in Problem 20 of Section 12.5 (page 735).

4 Evaluate double or triple integrals

(13) To find

$$\int_0^1 \int_0^x \int_x^{x+y} (x + z) dz dy dx$$

we use the command

```
Integrate[x+z, {x,0,1}, {y,0,x}, {z,x,x+y}]
```

Notice the order of integration in the command.

(14) Do Problem 3 of Section 12.5 (page 734).

(15) Do Problem 4 of Section 12.5 (page 734).

(16) You can also define the solid before integrating. For example, to find $\iiint_R y\,dV$ where R is the solid is given by

$$R = \{(x, y, z) : \ 0 \le y \le 1, \ y \le x \le 1, \ 0 \le z \le xy\}$$

we can do the following:

```
R=ImplicitRegion[0<=y<=1 && y<=x<=1 && 0<=z<=x*y,{x,y,z}]
Integrate[y, {x,y,z} \[Element] R]
```

(17) Do Problem 9 of Section 12.5 (page 734).

5 Find the Jacobian of a change of variables

Let (x, y) be the old variables and (u, v) be the new variables. The syntax to find the Jacobian matrix is

```
Grad[{x,y}, {u,v}]
```

(18) For example, let $x = uv$ and $y = u+v$. The Jacobian matrix and its determinant are computed as follows:

```
x = u*v
y = u+v
J = Grad[{x,y},{u,v}]
Det[J]
```

For a nicer rendition of J, write

```
MatrixForm[J]
```

If the change of variable is in 3D, where the old variables are (x, y, z) and the new variables are (u, v, w), the determinant matrix is

```
Grad[{x,y,z}, {u,v,w}]
```

(19) Do Problem 4 of Section 12.8 (page 755).

(20) Do Problem 5 of Section 12.8 (page 755).

6 Sketch the new region after a change of variables

Let (x, y) be the old variables and (u, v) be the new variables. Suppose the relation between them are given by $(u, v) = f(x, y)$, where f is a vector function. For example, if $f(x, y) = (x + y, x/y)$ then this means $u = x + y$ and $v = x/y$. The question is: what is the region R of (x, y) transformed into after the change of variables ?

(21) Let us consider a specific example where

$$R = \{(x, y) : \ x^2 + y^2 \leq 1, \ y \geq 0\}.$$

In other words, R is the upper half of the unit disk. Let $u = x + y$ and $v = x$. We want to find the region S of (u, v). S is called the *image* of R under the change of variables. First, define the region R as follows:

```
R = ImplicitRegion[x^2+y^2<=1 && y>=0, {x,y}]
```

Then define the change of variables:

```
f[x_,y_] := {x+y, x}
```

Then use the command `TransformedRegion` to get the region S.

```
S = TransformedRegion[R, f]
```

Then plot S.

```
Region[S, Axes->True, AspectRatio->Automatic, AxesLabel->{u,v}]
```

(22) Draw the image of the region bounded by the parabola $y = x^2$ and the line $y = 4$ under the change of variables $u = x - y$, $v = x^2 - 2y$.

(23) Draw the image of the unit ball (i.e. the inside of the unit sphere) under the change of variables $u = x$, $v = x - y$, $w = z^2$.

7 To turn in

Submit your implementation of Exercises (1) - (23) as a single pdf file.