Lecture 17

Tuesday, February 7, 2023 12:22 PM

* Questions ...

Chain rule.

$$\left.\begin{aligned} z &= f(x,y) \\ x &= x(t) \\ y &= y(t) \end{aligned}\right\} \quad z = z(t)$$

$$t \to t + dt$$

$$x \to x + dx = x + x'dt$$

$$y \to y + dy = y + y'dt$$

$$z \to z + dz = z + f_x dx + f_y dy$$

Thus, $dz = f_x dx + f_y dy = (f_x x' + f_y y')dt$

$$\frac{dz}{dt} = f_x x' + f_y y' = \underbrace{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}_{\text{chain rule}}$$

Imagine, if z only depends on x, not y, then $\frac{\partial f}{\partial y} = 0$ and we get

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt}$$

which is the familiar chain rule.

Ex: $x = 2t + 1$

$y = t^2 + t$

$z = x^2 + xy + y^2$

what is $z'(1)$?

Two ways

(1) $z = x^2 + xy + y^2 = (2t+1)^2 + (2t+1)(t^2+t) + (t^2+t)^2$

$$\frac{dz}{dt} = 2(2)(2t+1) + 2(t^2+t) + (2t+1)(2t+1) + 2(2t+1)(t^2+t)$$

$$z'(1) = 2(2)(3) + 2(2) + (3)(3) + 2(3)(2) = 37$$

(2) $\displaystyle \frac{dz}{dt} = z_x x' + z_y y' = (2x+y)2 + (2y+x)(2t+1)$

when $t=1$, $x = 2(1)+1 = 3$ and $y = 1^2+1 = 2$

$$\frac{dz}{dt}(1) = (2(3)+2)2 + (2(2)+3)(2(1)+1) = 16 + 21 = 37$$

More general chain rule

$$\left.\begin{array}{l} z = f(x,y) \\ x = x(u,v) \\ y = y(u,v) \end{array}\right\} \quad z = z(u,v)$$

$z_u = f_x x_u + f_y y_u$ or $\displaystyle \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

$z_v = f_x x_v + f_y y_v$ or $\displaystyle \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

Note. $z = f(x,y)$ is sometimes also written as $z = z(x,y)$

z: output

x,y: input

f: function

$$x,y \longrightarrow \boxed{f} \longrightarrow z$$

Sometimes it is more convenient to not talk about the machine, but only the relation between the input and the output.

$$\underbrace{x,y}_{\text{sugar, chocolate}} \longrightarrow \underbrace{\boxed{f}}_{\text{kitkat company}} \longrightarrow \underbrace{z}_{\text{kitkat candy}}$$

candy : combination of different ingredients in a suitable way.

kitkat : the name of the company that makes candy.

$$\underbrace{x,y}_{\text{parts}} \longrightarrow \boxed{\text{Toyota}} \longrightarrow \underbrace{z}_{\text{car}}$$

$\underline{\underline{\text{Ex}}}$

$$z = e^{x} \ln y$$

$$x = u + v$$

$$y = uv$$

what is $\frac{\partial z}{\partial u}(1,2)$?

$$\frac{\partial z}{\partial u} = z_x x_u + z_y y_u = e^{x}(\ln y)1 + \frac{e^{x}}{y} v$$

When $u=1$ and $v=2$, $x=3$ and $y=2$.

$$\frac{\partial z}{\partial u}(1,2) = e^{3}\ln 2 + \frac{e^{3}}{2}2 = e^{3}\ln 2 + e^{3}.$$