

# Lecture 17

Tuesday, February 7, 2023

12:22 PM

\* Questions ---

Chain rule.

$$\left. \begin{array}{l} z = f(x, y) \\ x = x(t) \\ y = y(t) \end{array} \right\} z = z(t)$$

$$t \rightarrow t + dt$$

$$x \rightarrow x + dx = x + x' dt$$

$$y \rightarrow y + dy = y + y' dt$$

$$z \rightarrow z + dz = z + f_x dx + f_y dy$$

Thus,

$$dz = f_x dx + f_y dy = (f_x x' + f_y y') dt$$

$$\frac{dz}{dt} = f_x x' + f_y y' = \underbrace{\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}}_{\text{chain rule}}$$

Imagine, if  $z$  only depends on  $x$ , not  $y$ , then  $\frac{\partial f}{\partial y} = 0$  and we get

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt}$$

which is the familiar chain rule.

Ex.  $x = 2t + 1$

$$y = t^2 + t$$

$$z = x^2 + xy + y^2$$

What is  $z'(1)$ ?

## Two ways

$$(1) \quad z = x^2 + xy + y^2 = (2t+1)^2 + (2t+1)(t^2+t) + (t^2+t)^2$$

$$\frac{dz}{dt} = 2(2)(2t+1) + 2(t^2+t) + (2t+1)(2t+1) + 2(2t+1)(t^2+t)$$

$$z'(1) = 2(2)(3) + 2(2) + (3)(3) + 2(3)(2) = 37$$

$$(2) \quad \frac{dz}{dt} = z_x x' + z_y y' = (2x+y)2 + (2y+x)(2t+1)$$

$$\text{When } t=1, \quad x = 2(1)+1=3 \quad \text{and} \quad y = 1^2+1=2$$

$$\frac{dz}{dt}(1) = (2(3)+2)2 + (2(2)+3)(2(1)+1) = 16 + 21 = 37$$

## More general chain rule

$$\left. \begin{array}{l} z = f(x, y) \\ x = x(u, v) \\ y = y(u, v) \end{array} \right\} z = z(u, v)$$

$$z_u = f_x x_u + f_y y_u \quad \text{or} \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$z_v = f_x x_v + f_y y_v \quad \text{or} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

Note.  $z = f(x, y)$  is sometimes also written as  $z = z(x, y)$

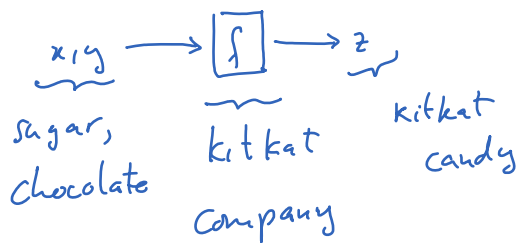
$z$ : output

$x, y$ : input

$f$ : function

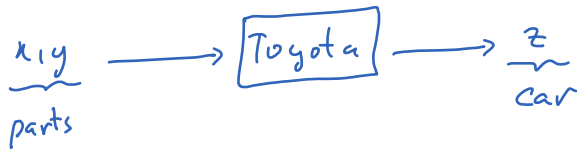
$$x, y \longrightarrow \boxed{f} \longrightarrow z$$

Sometimes it is more convenient to not talk about the machine, but only the relation between the input and the output.



candy: combination of different ingredients in a suitable way.

kitkat: the name of the company that makes candy.



$$z = e^x \ln y$$

$$x = u + v$$

$$y = uv$$

what is  $\frac{\partial z}{\partial u}(1, 2)$ ?

$$\frac{\partial z}{\partial u} = z_x x_u + z_y y_u = e^x (\ln y) 1 + \frac{e^x}{y} v$$

when  $u=1$  and  $v=2$ ,  $x=3$  and  $y=2$ .

$$\frac{\partial z}{\partial u}(1, 2) = e^3 \ln 2 + \frac{e^3}{2} 2 = e^3 \ln 2 + e^3.$$