Lecture 21

Thursday, February 16, 2023 12:11 PM

* Questions.

Optimization problem

$$f(x,y) \to \min/\max$$

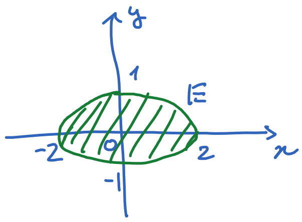
This problem depends on the function f as well as the region E where (x,y) is in.

Ex $f(x,y) = x^2+y^2$

$\min_{\mathbb{R}^2} f = 0$ attained at $(0,0)$

$\max_{\mathbb{R}^2} f = DNE$

Ex $f(x,y) = x^2+y^2$



$\min_E f = 0$, attained at $(0,0)$.

$\max_E f = 4$, attained at $(\pm 2, 0)$.

Ex $f(x,y) = x^3 + xy^2 - 6x + y^2$

$f(x,y) \to \min/\max$ on the disk of radius $\frac{3}{2}$ centered at the origin.

This is a tricker problem We will learn how to solve it step by step.

* Find the critical points of the function f

Solve $\nabla f(x,y) = 0$.

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \rightsquigarrow \begin{cases} 3x^2+y^2-6=0 \\ 2xy+2y=0 \rightsquigarrow 2y(x+1)=0 \rightsquigarrow y=0 \text{ or } x=-1 \end{cases}$$

* If $y=0$. substitute $y=0$ into the equation $3x^2+y^2-6=0$

$$3x^2-6=0 \rightsquigarrow x=\pm\sqrt{2}$$

We get two critical points: $(\pm\sqrt{2},0)$

* If $x=-1$: substitute $x=-1$ into $3x^2+y^2-6=0$

$$3+y^2-6=0 \rightsquigarrow y=\pm\sqrt{3}$$

We get two more critical points. $(-1,\pm\sqrt{3})$

In conclusion, there are four critical points: $(\pm\sqrt{2},0)$, $(-1,\pm\sqrt{3})$.

Use Mathematica to visualize these critical points. We observe that $(-\sqrt{2},0)$ is where f attains local maximum; $(\sqrt{2},0)$ is where f attains local minimum; $(-1,\pm\sqrt{3})$ are the saddle points.

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f[x_,y_]:= x^3 + xy^2 - 6x + y^2
ContourPlot[ f[x,y], {x,-2,2}, {y,-3,3}, Contours→100}
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<u>Second derivative Test</u>:

$$D=f_{xx}f_{yy}-f_{xy}^2$$

- If $D>0$ and $f_{xx}>0$ then (x,y) is a local min
- If $D>0$ and $f_{xx}<0$ then (x,y) is a local max.
- If $D<0$ then (x,y) is a saddle point.

<u>Ex</u> $f_{xx}=6x$

$f_{yy}=2(x+1)$

$f_{xy}=2y$

At $(x,y) = (\sqrt{2}, 0)$,

$$f_{xx} = 6\sqrt{2} > 0$$

$$f_{yy} = 2(\sqrt{2}+1) > 0$$

$$f_{xy} = 0$$

$$D = f_{xx} f_{yy} - f_{xy}^2 = 6\sqrt{2}\, 2(\sqrt{2}+1) > 0$$

Therefore, $(\sqrt{2}, 0)$ is where f attains local min