Lecture 22

Friday, February 17, 2023 12:19 PM

* Questions ...

Classification of critical points

$$\nabla f(x_0, y_0) = 0$$

- Compute $f_{xx}(x_0,y_0)$, $f_{xy}(x_0,y_0)$, $f_{yy}(x_0,y_0)$.
- Compute $D = f_{xx}f_{yy} - f_{xy}^2$.
  - If $D>0$ and $f_{xx}>0$ then (x,y) is a local min
  - If $D>0$ and $f_{xx}<0$ then (x,y) is a local max.
  - If $D<0$ then (x,y) is a saddle point.

To find min/max of a function in a region, we don't need to classify the critical point

$$f(x,y) \to \text{min/max when } (x,y) \in E$$

<u>Procedure</u> (analogous to Calc I)

- Find all critical points of f on E.
- Find min/max of f on the boundary of E.
- Compare the value of f at the critical points and the min/max of f on the boundary. The largest of those is $\max_E f$. The smallest is $\min_E f$.

Ex $f(x,y) = x^3 + xy^2 - 6x + y^2 \to$ min/max

when $(x,y)\in E$, the disk centered at the origin with radius $3/2$.

Step 1. find the critical points in E

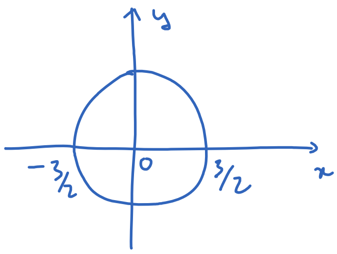
From last class, we know that there are 4 critical points.

$$(\pm\sqrt{2},0),\ (-1,\pm\sqrt{3})$$

Only $(\pm\sqrt{2},0)$ lie in the disk.

$f(\sqrt{2},0) = -4\sqrt{2}$, $\quad f(-\sqrt{2},0) = 4\sqrt{2}$.

Step 2: Find min/max on the circle



$$x^2+y^2=\left(\frac{3}{2}\right)^2=\frac{9}{4} \rightsquigarrow y^2=\frac{9}{4}-x^2$$

$$\begin{aligned} f(x,y) &= x^3+xy^2-6x+y^2 \\ &= x^3+x\left(\frac{9}{4}-x^2\right)-6x+\frac{9}{4}-x^2 \\ &= -x^2-\frac{15}{4}x+\frac{9}{4} = g(x) \end{aligned}$$

We want to find min/max of $g(x)$ when $x\in\left[-\frac{3}{2},\frac{3}{2}\right]$.

$$g'(x) = -2x-\frac{15}{4}$$

$$g'(x)=0 \quad \text{if} \quad x=-\frac{15}{8}\notin\left[-\frac{3}{2},\frac{3}{2}\right]$$

$$\min_{[-\frac{3}{2},\frac{3}{2}]} g = \min\left\{g\left(\frac{3}{2}\right), g\left(-\frac{3}{2}\right)\right\} = g\left(\frac{3}{2}\right) = -\frac{45}{8}$$

$$\max_{[-\frac{3}{2},\frac{3}{2}]} g = g\left(-\frac{3}{2}\right) = \frac{45}{8}$$

Therefore,

$$\min_{\text{b.d.}} f = f\left(\frac{3}{2},0\right) = -\frac{45}{8}$$

$$\max_{\text{b.d.}} f = f\left(-\frac{3}{2},0\right) = \frac{45}{8}$$

<u>Step 3</u>. Compare $\pm\frac{45}{8}$, $\pm 4\sqrt{2}$

$$\max_E f = f(-\sqrt{2}, 0) = 4\sqrt{2}$$

$$\min_E f = f(\sqrt{2}, 0) = -4\sqrt{2}$$