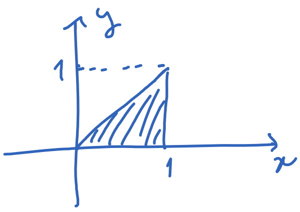
Lecture 26

Friday, February 24, 2023 12:13 PM

* Questions...

Double integral over a general region.

$\iint_D \sin(x^2)\, dA$ where D is the triangle below.



D has two descriptions:

① $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$

② $D = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\}$

One description will make the integral easier

If we use the second description then

$$\iint_D \sin(x^2)\, dA = \int_0^1 \underbrace{\int_y^1 \sin(x^2)\, dx\, dy}$$

stuck because we don't know what an antiderivative of $\sin(x^2)$ is.

If we use the first description then

$$\iint_D \sin(x^2)\, dA = \int_0^1 \int_0^x \sin(x^2)\, dy\, dx = \int_0^1 x \sin(x^2)\, dx$$

Let $u = x^2$. Then $du = 2x\,dx$ and $dx = \frac{du}{2x}$.

$$\int_0^1 x \sin(x^2)\, dx = \int_0^1 x \sin u \frac{du}{2x} = \frac{1}{2}\int_0^1 \sin u\, du = \frac{1}{2}(-\cos u)\Big|_0^1 = \frac{1}{2} - \frac{1}{2}\cos 1$$

* Moral of the example: if you can't solve a double integral, find a

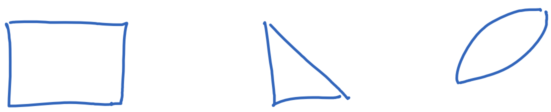
different way to describe the region

In Calc I, the difficulty of the integral $\int_a^b f(x)\,dx$ lies in the function f.

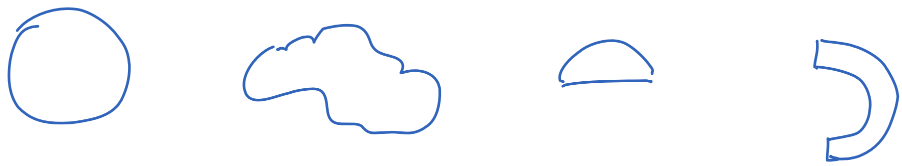
If the function is complicated, the integral is difficult. If the function is simple, the integral is also simple.

For multivariable Calculus, it is usually the region D that causes difficulty, not the function f

* Easy regions:

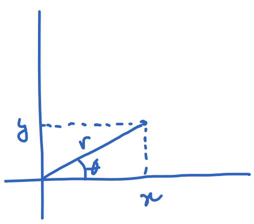


* Tricky regions



We will consider a method to find double integral over "circle" regions.

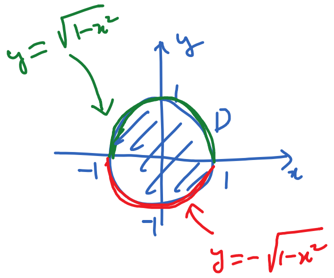
* Polar coordinates:



$x = r\cos\theta\ ,\quad y = r\sin\theta$

(x, y) : Cartesian coords.

(r, θ) : polar coords.

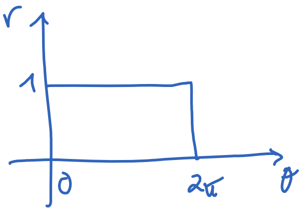


$$\iint_D x^2\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2\,dy\,dx$$

$$= 2\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx \rightsquigarrow \text{tricky!}$$

D has a nice description (free of square roots) in polar coordinates.

$$\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$



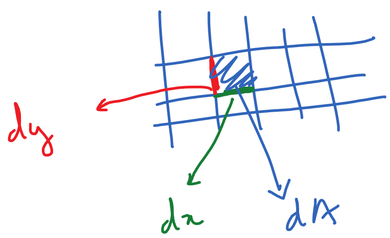
D is called a polar rectangle because it is a rectangle in terms of the polar coordinates.

$$\iint_D x^2 \, dA$$

$$x^2 = r^2\cos^2\theta$$

$$dA = ??$$

We know that $dA = dx\,dy$. But we want to express dA in terms of dr and $d\theta$.



$$\underbrace{dA}_{\text{area}} \neq \underbrace{dr}_{\text{length}}\,\underbrace{d\theta}_{\text{dimensionless}}$$

$$\underbrace{dA}_{\substack{\text{small}\\ \text{of order 2}}} = \underbrace{dr\,dr\,d\theta}_{\text{small of order 3}}$$

<u>Answer</u>. $dA = r\,dr\,d\theta$ (to be explained next time)

* <u>Ex</u>

$$\iint_D x^2\,dA = \int_0^{2\pi}\int_0^1 r^2\cos^2\theta \; r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4}\cos^2\theta \Big|_{r=0}^{r=1} d\theta = \frac{1}{4}\int_0^{2\pi}\cos^2\theta\,d\theta$$

Double angle identity:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\iint_D x^2\,dA = \frac{1}{8}\int_0^{2\pi}(1+\cos(2\theta))\,d\theta = \frac{1}{8}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = \frac{2\pi}{8} = \frac{\pi}{4}.$$