Lecture 28

Tuesday, February 28, 2023 11:54 AM

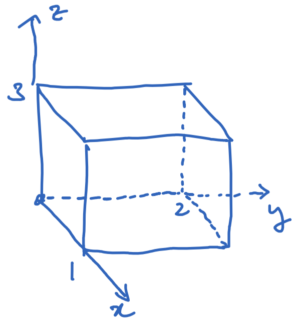
* Questions..

Triple integral:

If E is a rectangular box $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ then

$$\iiint_E f(x,y,z)\,dV = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)\,dz\,dy\,dx$$

Ex



mass density: $f(x,y,z) = x+y+z$

$$\text{Total mass} = \int_0^1\int_0^2\int_0^3 (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^2 \left(xz + yz + \frac{z^2}{2}\right)\Big|_{z=0}^{z=3} dy\,dx$$

$$= \int_0^1\int_0^2 \left(3x + 3y + \frac{9}{2}\right) dy\,dx$$

$$= \int_0^1 \left(3xy + \frac{3y^2}{2} + \frac{9}{2}y\right)\Big|_0^2 dx$$

$$= \int_0^1 (6x + 6 + 9)\,dx$$

$$= \int_0^1 (6x+15)\,dx = (3x^2 + 15x)\Big|_0^1 = 18$$

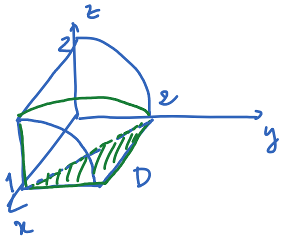
A more general solid is of the form

$$E = \{(x,y,z) : g(x,y) \le z \le h(x,y),\ (x,y) \in D\}$$

In this case,

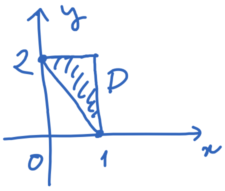
$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz\,dA$$

Ex



Mass density: $f(x,y,z) = yz$

$E = \{(x,y,z) \quad 0 \le z \le \sqrt{4-y^2},\ (x,y) \in D\}$



$$\text{Total mass} = \iint_D \int_0^{\sqrt{4-y^2}} yz\,dz\,dA$$

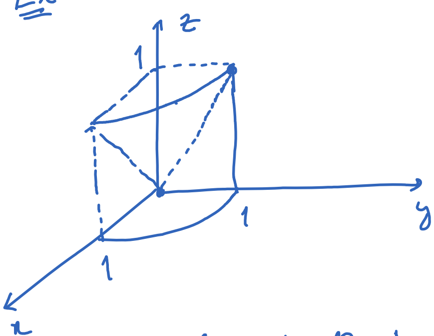
$$= \frac{1}{2}\iint_D y(4-y^2)\,dA = \frac{1}{2}\int_0^2 \int_{1-\frac{y}{2}}^{1} y(4-y^2)\,dx\,dy$$

$$= \frac{1}{2}\int_0^2 \underbrace{\left(1-\frac{y}{2}\right)y(4-y^2)}_{\left(y-\frac{y^2}{2}\right)(4-y^2)}\,dy$$

$$= 4y - 3y^2 + \frac{y^4}{2}$$

$$= \frac{1}{2}\left(2y^2 - y^3 + \frac{y^5}{10}\right)\Big|_0^2 = \frac{1}{2}\left(8 - 8 + \frac{32}{10}\right) = \frac{8}{5}$$

Ex



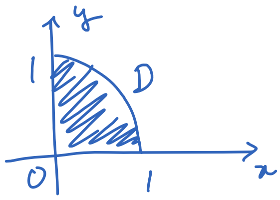
$$\iiint_E (x^2+y^2)\,dV = ?$$

$E = \{(x,y,z) . \quad x+y \le z \le 1,\ (x,y) \in D\}$

0, 1, 1, 0, 1

1, 0, 1, 1, 0

1, 1, −1



$$\iiint_E (x^2+y^2)\,dV = \iint_D \int_{x+y}^{1} (x^2+y^2)\,dz\,dA = \iint_D (x^2+y^2)(1-x-y)\,dA$$

$$= \int_0^{\pi/2}\int_0^1 r^2(1-r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 (r^2 - r^3\cos\theta - r^3\sin\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3} - \frac{r^4}{4}\cos\theta - \frac{r^4}{4}\sin\theta\right)\Big|_0^1\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{3} - \frac{1}{4}\cos\theta - \frac{1}{4}\sin\theta\right)d\theta$$

$$= \left(\frac{\theta}{3} - \frac{1}{4}\sin\theta + \frac{1}{4}\cos\theta\right)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{6} - \frac{1}{4} - \left(\frac{1}{4}\right) = \frac{\pi}{6} - \frac{1}{2}.$$