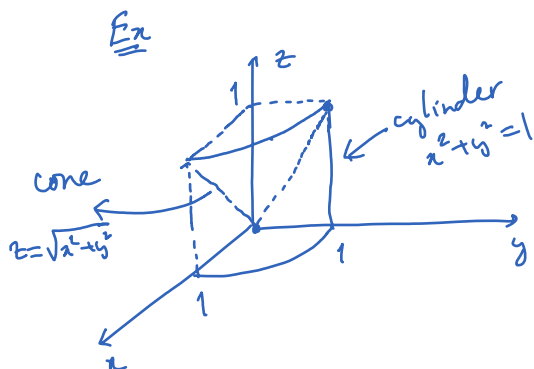


Lecture 29

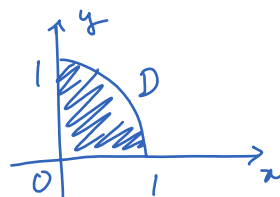
Thursday, March 2, 2023 12:01 PM

* Questions ..



$$\iiint_E (x^2 + y^2) dV = ?$$

$$E = \{(x, y, z) \cdot \sqrt{x^2 + y^2} \leq z \leq 1, (x, y) \in D\}$$



$$\iiint_E (x^2 + y^2) dV = \iint_D \int_{\sqrt{x^2 + y^2}}^1 (x^2 + y^2) dz dA = \iint_D (x^2 + y^2)(1 - \sqrt{x^2 + y^2}) dA$$

$$= \int_0^{\pi/2} \int_0^1 r^2(1-r) r dr d\theta$$

$$= \int_0^{\pi/2} \int_0^1 (r^3 - r^4) dr d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4} - \frac{r^5}{5} \right) \Big|_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{20} d\theta = \frac{\pi}{40}$$

* Change of variables

In Calc I $\int_a^b f(x) dx$

Change of variable is needed when the function f is complicated.

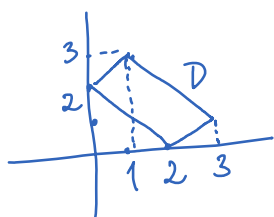
In Calc IV, change of variable is needed mostly because the region is complicated (i.e. not rectangular).

$$\iint_D f(x,y) dA = \iint_{D'} f(x(u,v), y(u,v)) dA'$$

$$\frac{dA}{dA'} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \underbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right|}_{\text{Jacobian of the transformation}} dA'$$

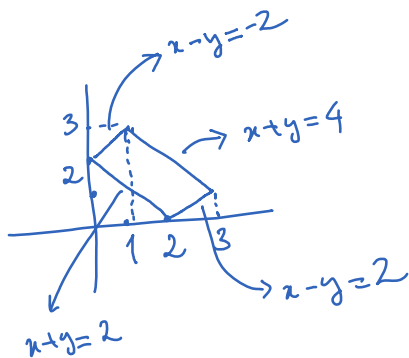
Here $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$
 Jacobian matrix

Ex



D = the parallelogram with vertices at $(2,0), (3,1), (1,3), (0,2)$.

$$\iint_D (x+y) dA = ?$$

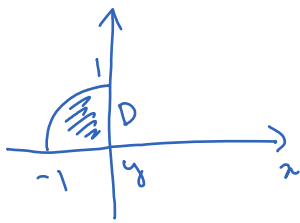


Let $u=x+y$ and $v=x-y$. Then $2 \leq u \leq 4, -2 \leq v \leq 2$

$$\begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -1/2$$

$$\iint_D (x+y) dA = \iint_{D'} u \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA'$$

$$= \iint_D u \frac{1}{2} dA' = \int_2^4 \int_{-2}^2 \frac{u}{2} dv du = \int_2^4 2u du = u^2 \Big|_2^4 = 12$$



$$\iint_D (y-x) dA$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\underbrace{0 \leq r \leq 1, \frac{\pi}{2} \leq \theta \leq \pi}_{D'}$$

$$x_r = \cos \theta, \quad x_\theta = -r \sin \theta$$

$$y_r = \sin \theta, \quad y_\theta = r \cos \theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

Thus,

$$\iint_D (y-x) dA = \iint_{D'} (r \sin \theta - r \cos \theta) r dA'$$

$$= \int_{\pi/2}^{\pi} \int_0^1 (r^2 \sin \theta - r^2 \cos \theta) dr d\theta$$

$$= \int_{\pi/2}^{\pi} \frac{\sin \theta - \cos \theta}{3} d\theta = \left. \frac{-\cos \theta - \sin \theta}{3} \right|_{\pi/2}^{\pi} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$