Lecture 30

Thursday, March 2, 2023 10:57 PM

* Questions..

In Calc IV, change of variable is needed mostly because the region is complicated (i.e. not rectangular).

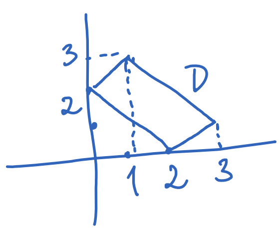
$$\iint_D f(x,y)\,dA = \iint_D f(x(u,v), y(u,v))\,dA$$

$$= \iint_{D'} f(x(u,v), y(u,v)) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{Jacobian of the transformation}} dA'$$

$$\frac{dA}{dA'} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

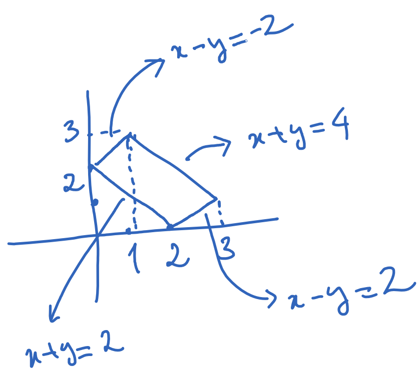
Here $\dfrac{\partial(x,y)}{\partial(u,v)} = \det \underbrace{\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}}_{\text{Jacobian matrix}}$

Ex



D = the parallelogram with vertices at $(2,0), (3,1), (1,3), (0,2)$.

$$\iint_D (x+y)\,dA = ?$$



Let $u = x+y$ and $v = x-y$. Then

$2 \le u \le 4$, $-2 \le v \le 2$

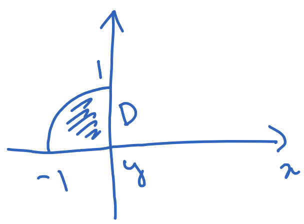
$$\begin{cases} x = \dfrac{u+v}{2} \\[2ex] y = \dfrac{u-v}{2} \end{cases} \rightsquigarrow \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$= -1/2$$

$$\iint_D (x+y)\,dA = \iint_{D'} u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA'$$

$$= \iint_D u \tfrac{1}{2}\, dA' = \int_2^4 \int_{-2}^{2} \frac{u}{2}\, dv\, du = \int_2^4 2u\, du = u^2\Big|_2^4 = 12$$

Ex



$$\iint_D (y-x)\,dA$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \underbrace{0 \le r \le 1, \quad \frac{\pi}{2} \le \theta \le \pi}_{D'}$$

$x_r = \cos\theta$, $\quad x_\theta = -r\sin\theta$

$y_r = \sin\theta$, $\quad y_\theta = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Thus,

$$\iint_D (y-x)\,dA = \iint_{D'} (r\sin\theta - r\cos\theta)\, r\, dA'$$

$$= \int_{\pi/2}^{\pi} \int_0^1 (r^2\sin\theta - r^2\cos\theta)\, dr\, d\theta$$

$$= \int_{\pi/2}^{\pi} \frac{\sin\theta - \cos\theta}{3}\, d\theta = \left. \frac{-\cos\theta - \sin\theta}{3} \right|_{\pi/2}^{\pi} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$