Lab 6

In this lab, you will practice with sequences, series, and power series on Mathematica. Specifically, you will learn how to

- Compute limit of a sequence,
- Compute partial sums and series,
- Find the radius of convergence and interval of convergence of a power series,
- Use power series to solve differential equations,
- Approximate a function by polynomials.

1 To turn in

Do Problems 1-27 in a single Mathematica Notebook file, or ipynb file if you use JupyterLab. Write *your name* and *lab number* at the beginning of your report. Clearly label each problem to separate them from other problems. Make sure to comment on each problem. If your code doesn't run correctly, explain what you are trying to do. **Failed code without any comment/explanation will receive 0 point.** Submit on Canvas both the *pdf file* and the *source file* (nb or ipynb).

Problems	Points
3, 8, 13-15, 23	1
1, 2, 4-6, 9-12, 18, 22, 25, 26	2
7, 17, 19, 24, 27	3
17, 20, 21	4
Readability of your report	3
Total: 27	Total: 62

2 Define and evaluate terms and limit of a sequence

1. A sequence is a function from $\mathbb{N}$ to $\mathbb{R}$. Thus, it can be defined the same way you would define a function. For example, the sequence $a_n = \frac{(-1)^{n+1}}{2^n}$ can be defined as follows.

   ```
   a[n_]:=(-1)^(n+1)/2^n
   ```

   To view the first 20 terms of this sequence, try the command

   ```
   Table[{n, a[n]}, {n, 1, 20}] // TableForm
   ```

   To convert the fractions into decimal-point numbers, replace `a[n]` in the command by `N[a[n]]`. If you want 6 significant digits, then use `N[a[n],6]` instead. Use the command

   ```
   Limit[a[n], n -> Infinity]
   ```

   to find the limit of this sequence.

2. Find the first 30 terms of the sequence

$$a_n = \left(\frac{n^2+n}{n^2+1}\right)^n$$

and find the limit of the sequence (exactly and numerically).

3. Because a sequence is also a function, you can graph a sequence. However, since the variable n is a natural number, you will use `DiscretePlot` instead of `Plot`.

```
DiscretePlot[a[n], {n,1,20}, PlotRange -> Full]
```

The horizontal axis shows the indices n and the vertical axis shows the values of a_n.

4. Graph the sequence $a_n = \sin n$ for $n = 0, 1, 2, ..., 100$. Does the sequence converge? Why?

5. Graph the sequence

$$a_n = \frac{2n^2 + n\sin(n)}{n^2+1}$$

Find the exact value of the limit of a_n. Does the graph confirm the result you found?

6. A sequence can also be defined by a recursive formula. For example, the Fibonacci sequence $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ can be defined in Mathematica as

```
a[1] := 1
a[2] := 1
a[n_] := a[n - 1] + a[n - 2]
```

Find the first 20 terms of this sequence.

7. Find the first 20 terms of the sequence defined recursively as

$$a_0 = a_1 = a_2 = 1, \quad a_n = a_{n-1} - 2a_{n-2} + a_{n-3}$$

Graph the sequence using `DiscretePlot`. Does the sequence appear to converge or diverge?

The command `Limit` doesn't work well with sequences that are defined recursively. To find the limit of such a sequence, we use the command `RSolveValue` instead. The syntax is as follows:

```
RSolveValue[eqn, expr, n]
```

where `eqn` is a list consisting of the recursive formula and all initial conditions, `expr` is whatever expression you want to compute, and `n` is the index variable.

8. For example, we want to find the limit of the sequence

$$a_{n+1} = \frac{a_n + na_{n-1}}{n+1}, \quad a_0 = 0, \; a_1 = 1.$$

Symbolically, the limit of a_n may be viewed as a_∞ (as if n is substituted by ∞). Thus, a_∞ is the value we want to find. Try the following:

```
Clear[a]
RSolveValue[{a[n + 1] == (a[n] + n a[n - 1])/(n + 1),
            a[0] == 0, a[1] == 1}, a[Infinity], n]
```

9. Consider a sequence defined recursively as follows:

$$a_{n+1} = \frac{2a_n + 3}{a_n + 4}, \quad a_0 = -2.$$

Find the limit of the sequence using `RSolveValue`. Then graph the sequence to confirm the result.

10. Consider the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

To define the k'th partial sum, do the following:

```
a[n_] := Log[n]/n^2;
s[k_] := Sum[a[n], {n, 1, k}];
```

To get an idea on whether the partial sums converge, find the first 50 (or more) partial sums using the command `Table` and graph them using `DiscretePlot`. From the table and graph, can you guess if the series $\sum a_n$ converges or diverges?

11. You can find the exact value of the series by

```
Sum[a[n], {n,1,Infinity}]
```

Change `Sum` to `N[Sum[...],10]` for an approximate value with 10 significant digits. Check if $\sum_{n=1}^{\infty} a_n = \lim s_k$.

12. Find the value of the following series correct to 8 decimal places (if it converges)

$$\sum_{n=1}^{\infty} \frac{n^2 - 3n \sin n}{n^4 + 1}$$

3 Find radius and interval of convergence

Recall that a *power series centered at a* is a series of the form $\sum c_n(x-a)^n$. On its interval of convergence, a power series defines a function. Not counting the endpoints, the interval of convergence is always a symmetric interval about the center of the power series. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

To find the radius of convergence, one can use either the Ratio Test or the Root Test. Let

$$a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$$

The Ratio Test says that if the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

exists and $L < 1$ then the series converges. If $L > 1$ then the series diverges. If $L = 1$ then the test fails.

13. We compute the limit L as follows.

```
a[n_] := (-3)^n*x^n/Sqrt[n + 1]
Clear[n]
L = Limit[Abs[a[n + 1]/a[n]], n -> Infinity]
```

Here, `Abs` is the absolute value function.

14. The values of x that makes $L < 1$ belongs to the interval of convergence. We solve the inequality $L < 1$ as follows.

```
Reduce[L < 1, x, Reals]
```

The option `Reals` in the above command is to indicate that we are interested in x as a real number (as opposed to complex number).

15. You will see that the inequality $L < 1$ gives $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$. The radius of convergence is a half of the length of this interval, which is $R = \frac{1}{3}$. The endpoints $-1/3$ and $1/3$ have to be considered manually and separately. Mathematica can provide some insights as follows. Let $f(x)$ denote the value of the power series.

```
f[x_] := Sum[(-3)^n*x^n/Sqrt[n + 1], {n, 0, Infinity}]
```

We can attempt to evaluate f at $x = 1/3$ and $x = -1/3$.

```
f[1/3]
f[-1/3]
```

Mathematica will show a warning on the second command, indicating that f is not defined at $x = -1/3$. Therefore, the interval of convergence is $(-1/3, 1/3]$.

16. Find the center, the radius of convergence, and the interval of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Then approximate the value of the power series at $x = 0$.

4 Solve differential equation using power series

Consider the initial value problem

$$(1+x)y' = \frac{1}{2}y, \qquad y(0) = 1$$

This problem can be solved using separation of variables method or integrating factor method. Alternately, you can use power series to solve it (at least approximately). Because the initial value of x is 0, you will represent y as a power series centered at 0. That is, $y = \sum_{n=0}^{\infty} a_n x^n$ where the coefficients $a_0, a_1, a_2, ...$ are to be found. A computer cannot find for you infinitely many coefficients, so you need to truncate this series. For example,

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^5)$$

Here, $O(x^5)$ (letter O, not number 0) denotes the tail of the series that has been cut off, which consists of terms of order x^5 or higher.

17. Run the following commands (letter O, not number 0)

```
Clear[a]
y[x_] := Sum[a[n]*x^n, {n, 0, 4}] + O[x]^5
(1 + x) y'[x] - 1/2 y[x]
```

The last expression is supposed to be 0. That means the coefficients of $x, x^2, x^3, ...$ are supposed to be 0. The initial condition $y(0) = 1$ implies that $a_0 = 1$. Find a_1, a_2, a_3, a_4.

18. Use the command `DSolve` to solve for the exact solution of the initial value problem. Review Lab 4 if you forget this command.

19. Graph this exact solution together with the approximate solution $y \approx a_0 + a_1 x + ... + a_4 x^4$, with coefficients a_1, a_2, a_3, a_4 found earlier, on the same plot. You should plot on a narrow interval of x centered at 0. How close is the approximate solution compared to the exact solution?

20. Repeat Exercises 17-19 but now include 3 more terms in the series: $y = a_0 + a_1 x + ... + a_7 x^7 + O(x^8)$. Does including more terms in the series give you a better approximation of the solution?

21. Both separation of variables method and integrating factor method fail to solve the initial value problem

$$y' = x + y^2, \quad y(0) = -1.$$

Use power series to solve it.

5 Approximate a function by polynomials

In many applications, it is helpful to approximate a function with polynomials. Polynomials are easier to take derivative or integral, and more computer-friendly because they involve only the addition, subtraction, and multiplication. To approximate a function by a polynomial, we simply truncate the Taylor series of the function. The Taylor series centered at a of a function f is *defined* by

$$\sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}$$

Truncating this series at a power m, we get an m'th degree Taylor polynomial

$$T_m(x) = \sum_{n=0}^{m} c_n (x-a)^n$$

and f is approximated by $f(x) \approx T_m(x)$. The approximation is good when x is close to a, which is the center of the power series, and gets worse as x is far away from the center. To maintain a good approximation when x is far away from a, you will have to increase the degree m. The larger m is, the farther away x can be from a and the approximation is still good.

To obtain the Taylor polynomials T_m centered at $a = 0$ of a function f, we use the command **Series** with the syntax

```
Series[f[x], {x, 0, m}]
```

22. For example, consider the function $f(x) = \sin x + \cos(x/\sqrt{2})$. Try the command

```
f[x_] := Sin[x] + Cos[x/Sqrt[2]]
Series[f[x], {x, 0, 7}]
```

The output will look something like

$$1 + x - \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{96} + \frac{x^5}{120} - \frac{x^6}{5760} - \frac{x^7}{5040} + O\left(x^8\right)$$

Therefore, the first seven Taylor polynomials $T_1, T_2, ..., T_7$ are

$$\begin{aligned}
T_1(x) &= 1 + x \\
T_2(x) &= 1 + x - \frac{x^2}{4} \\
T_3(x) &= 1 + x - \frac{x^2}{4} - \frac{x^3}{6} \\
&\dots \\
T_7(x) &= 1 + x - \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{96} + \frac{x^5}{120} - \frac{x^6}{5760} - \frac{x^7}{5040}
\end{aligned}$$

The term $O(x^8)$ is the error term (the difference between the function f and the polynomial T_7), consisting of terms of order 8 or higher.

23. To see how well each Taylor polynomial approximates the function f, we graph them together on the same plot. For example, try the following to graph f and T_1 on the same plot.

```
T1[x_] := 1+x
Plot[{f[x], T1[x]}, {x,-3,3}]
```

24. Use the fashion above to graph each of the functions $T_2, T_3, ..., T_7$ (one by one, not all at once) together with f on the same plot. What do you observe?

25. One way to quantify how good the approximation $f(x) \approx T_m(x)$ is on the interval $x \in [-3, 3]$ is by looking at the maximum value of $|f(x) - T_m(x)|$ on the interval $[-3, 3]$. Try

```
MaxValue[{Abs[f[x] - T1[x]], -3 <= x <= 3}, x]
NMaxValue[{Abs[f[x] - T1[x]], -3 <= x <= 3}, x]
```

26. For $m = 2, 3, ..., 7$, find the maximum of $|f(x) - T_m(x)|$ on the interval $[-3, 3]$.

27. How large does m have to be so that $f(x) \approx T_m(x)$ with an error less than 0.01 for any $x \in [-3, 3]$?