Lecture 12

Thursday, January 30, 2025 11:48 AM

Last time, we saw 4 numerical methods to evaluate numerically an integral $\int_a^b f(x)dx$. They are the left-point, right-point, midpoint, and trapezoid methods.

- $L_n = \frac{b-a}{n}\sum_{i=1}^{n} f(x_{i-1}) = \frac{b-a}{n}\left(f(x_0) + \cdots + f(x_{n-1})\right)$
- $R_n = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i) = \frac{b-a}{n}\left(f(x_1) + \cdots + f(x_n)\right)$
- $M_n = \frac{b-a}{n}\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) = \frac{b-a}{n}\left(f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right)$
- $T_n = \frac{b-a}{n}\sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}$
  $= \frac{b-a}{n}\left(\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right)$

For a given value of n, it takes about the same effort to compute L_n, R_n, M_n, T_n (about $n-1$ additions and 1 multiplication). Which method is the best to use?
The best method is the method that can give us an answer within an allowable error with the smallest value of n. In other words, the best method is the one that requires the least amount of computation to achieve the same precision goal.

We will estimate how large the difference is $\int_a^b f(x)dx - L_n$. You can expect that it gets smaller as n increases. It is natural to ask how large $\int_a^b f(x)dx - L_n$ is compared to $1/n$.

$$\int_a^b f(x)dx - L_n = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left(f(x) - f(x_{i-1})\right)dx$$

By Mean Value Theorem,

$$\frac{f(x) - f(x_{i-1})}{x - x_{i-1}} = f'(c)$$

for some $c \in (x_{i-1}, x)$. Thus,

$$|f(x) - f(x_{i-1})| = |f'(c)||x - x_{i-1}| \le \max_{[a,b]}|f'|\frac{b-a}{n}$$

Then

$$\left|\int_a^b f(x)dx - L_n\right| \le \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}|f(x) - f(x_{i-1})|dx$$
$$\le \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\max_{[a,b]}|f'|\frac{b-a}{n}dx = \max_{[a,b]}|f'|\frac{(b-a)^2}{n}$$

This estimate implies that the difference $\int_a^b f(x)dx - L_n$ is of the order $1/n$ as n increases. Skipping some computational details, we have

$$\left|\int_a^b f(x)dx - L_n\right| \leq \frac{C_1}{n}$$
$$\left|\int_a^b f(x)dx - R_n\right| \leq \frac{C_2}{n}$$
$$\left|\int_a^b f(x)dx - M_n\right| \leq \frac{C_3}{n^2}$$
$$\left|\int_a^b f(x)dx - T_n\right| \leq \frac{C_4}{n^2}$$

With the same value of n (sufficiently large), the midpoint and trapezoid methods yield smaller errors than the left-point and right-point methods.