Lecture 15

Wednesday, February 5, 2025 12:25 AM

Integral is not only used to compute area or volume, but also used to compute arc length. You can expect to see that integral and derivative blend together beautifully!

Consider the curve $y = f(x)$. We want to compute the length of the curve where $a \le x \le b$. Let $S(x)$ be the length of the portion of the curve above the interval $[a, x]$. Then

$$S(x+\Delta x) - S(x) \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta\text{x}$$

Dividing both sides by Δx and taking the limit as $\Delta\text{x} \to 0$, we get $S'(x) = \sqrt{1 + (y')^2}$.
Therefore,

$$\int_a^b \sqrt{1+(y')^2}\,dx = S(b) - S(a) = S(b)$$

Example: find the length of the curve $y = \frac{t^2}{8} - \ln t$ on the interval $t \in [1,4]$.
Notice that $y' = \frac{t}{4} - \frac{1}{t}$

$$1 + (y')^2 = 1 + \left(\frac{t}{4} - \frac{1}{t}\right)^2 = \frac{t^2}{16} + \frac{1}{t^2} + \frac{1}{2} = \left(\frac{t}{4} + \frac{1}{t}\right)^2$$

Thus, the length of the curve is

$$L = \int_1^4 \sqrt{1+(y')^2}\,dt = \int_1^4 \left(\frac{t}{4} + \frac{1}{t}\right) dt = \left(\frac{t^2}{8} + \ln t\right)\Bigg|_1^4 = \frac{15}{8} + \ln 4$$