Lecture 16

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Another example of finding arclength:
Find the length of the curve $y = \ln x$ on the interval $x \in [1, \sqrt{3}]$.
$y' = 1/x$
$1 + (y')^2 = 1 + \frac{1}{x^2} = \frac{x^2+1}{x^2}$
The length of the curve is

$$L = \int_1^{\sqrt{3}} \sqrt{\frac{x^2+1}{x^2}}\,dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x}\,dx$$

Let $x = \tan t$

x	1	$\sqrt{3}$
t	$\pi/4$	$\pi/3$

Then

$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan^2 t + 1}}{\tan t} \sec^2 t\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^3 t}{\tan t}\,dt$$

Method 1: use the substitution $u = \sec t$.

t	$\pi/4$	$\pi/3$
u	$\sqrt{2}$	2

Then

$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 t}{\tan^2 t} \sec t \tan t\,dt = \int_{\sqrt{2}}^{2} \frac{u^2}{u^2-1}\,du$$

This is an integral of a rational function, where the degree of the numerator is equal to the degree of the denominator. Use long division and then use partial fraction decomposition.

Method 2: convert $\tan t$ and $\sec t$ into $\cos t$ and $\sin t$.
$\frac{\sec^3 t}{\tan t} = \frac{1}{\cos^2 t \sin t} = \cos^{-2} t \sin^{-1} t$
Here, the power of sine is odd. We use the substitution $u = \cos t$.
$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^{-2} t \sin^{-2} t \sin t\,dt = \int_{1/2}^{\sqrt{2}/2} \frac{1}{u^2(1-u^2)}\,du$

Then use partial fraction decomposition.