Lecture 23

Thursday, February 20, 2025 1:30 AM

Consider the differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$.
Euler's idea to solve this equation is as follows.

- Discretize the x-axis by
  $$x_0 = x_0, x_1 = x_0 + h, x_2 = x_1 + h = x_0 + 2h, x_3 = x_2 + h = x_0 + 3h, \dots$$
- Plugging $x = x_0$ into the differential equation, we get $y'(x_0) = f(x_0, y_0)$.
- Use the approximation $y'(x_0) \approx \frac{y(x_1) - y(x_0)}{h}$, we get
  $y'(x_0) \approx y(x_0) + hy'(x_0) = y_0 + hf(x_0, y_0)$
  Let $y_1 = y_0 + hf(x_0, y_0)$.
- Plugging $x = x_1$ into the differential equation, we get $y'(x_1) = f\big(x_1, y(x_1)\big) \approx f(x_1, y_1)$.
- Use the approximation $y'(x_1) \approx \frac{y(x_2) - y(x_1)}{h}$, we get
  $y'(x_2) \approx y(x_1) + hy'(x_1) \approx y_1 + hf(x_1, y_1)$
  Let $y_2 = y_1 + hf(x_1, y_1)$.
- ...

In general, $y(x_n) \approx y_n$ where $y_0 = y_0$ and $y_{n+1} = y_n + hf(x_n, y_n)$.

Work on some examples on the worksheet.

At each position (a, b) on the plane, you can sketch a short line segment such that if the solution curve to the equation $y' = f(x, y)$ passes through (a, b), it must be tangent to this line segment. The collection of all these little line segments is called the *direction field* of the differential equation.