Lecture 28

Friday, February 28, 2025 9:50 AM

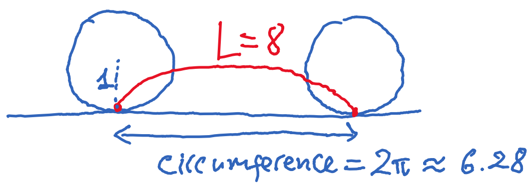
Recall the formula of the length of a parametric curve $x = x(t), y = y(t), a \le t \le b$

$$L = \int_a^b \sqrt{(x')^2 + (y')^2} dt$$

Example:

Find the length of the curve $x = t - \sin t\,, y = 1 - \cos t\,, 0 \le t \le 2\pi$

This is known as the cycloid curve. It is the trajectory of the valve of your bicycle with wheel radius 1.



$x' = 1 - \cos t\,, y' = \sin t$

$(x')^2 + (y')^2 = 2 - 2\cos t = 4\sin^2\left(\frac{t}{2}\right)$

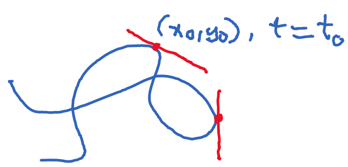
Thus,

$$L = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt = -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} = 8$$

In general, it is quite difficult to find exactly the length of a parametric curve due to the presence of the square root in the integrand.

What are other geometric properties of a graph? Tangent line and area.

We will use the same methodology as above to find out the respective formula for slope and area. Namely, we will start with the formula that we know for graph. And then adjust it to a form that is valid for any parametric curve.



Slope of the tangent line at a point (x_0, y_0) on the graph of the function $y = f(x)$ is $f'(x_0) = \frac{dy}{dx}(x_0)$. This formula requires y to be a function of x, which is not the case for most parametric curves. Here is the tweak:

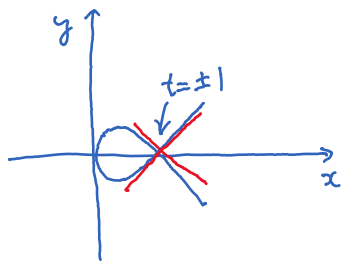
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$$

In this form, y isn't required to be a function of x. Instead, y and x are both functions of t. The equation of the tangent line is:

$$y = y_0 + \frac{y'(t_0)}{x'(t_0)}(x - x_0)$$

If the denominator is 0 then the tangent line is a vertical line with equation $x = x_0$.

Example:
From the last lecture, you know that the curve $x = t^2, y = t^3 - t$ intersects itself at $(x, y) = (1, 0)$ when $t = \pm 1$. Find the equations of the two tangent lines to the curve at this point.



$$\frac{y'}{x'} = \frac{3t^2 - 1}{2t}$$

The equation of the line when the curve crosses the point (1,0) the first time (when $t = -1$) is

$$y = y_0 + \frac{y'(t_0)}{x'(t_0)}(x - x_0) = 0 + (-1)(x - 1) = -x + 1$$

The equation of the line when the curve crosses the point (1,0) the second time (when $t = 1$) is

$$y = y_0 + \frac{y'(t_0)}{x'(t_0)}(x - x_0) = 0 + (1)(x - 1) = x - 1$$