Lecture 29

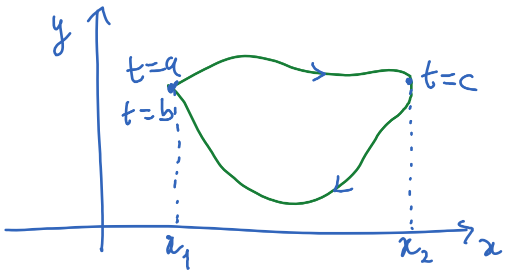
Monday, March 3, 2025 2:30 AM

A curve is *closed* if its ending point coincides its starting point. A curve is *simple* if it doesn't intersect itself.

The area enclosed by a closed simple curve $x = x(t), y = y(t), a \leq t \leq b$ is

$$A = \left|\int_a^b xy'dt\right| = \left|\int_a^b yx'dt\right|$$

A full proof of this result is little involved. It is a consequence of the *Green's theorem*, which you will learn in Math 314. However, you can explain it intuitively as follows.



Suppose the curve is a graph $y = y(x)$ when $a \leq t \leq c$ and when $c \leq t \leq b$. Let $x_1 = x(a)$ and $x_2 = x(b)$. The area under the upper curve $y = y(x)$, where $x_1 \leq x \leq x_2$ is

$$A_1 = \int_{x_1}^{x_2} ydx = \int_a^c yx'dt$$

The area under the lower curve $y = y(x)$, where $x_1 \leq x \leq x_2$ iI

$$A_2 = \int_{x_1}^{x_2} ydx = \int_b^c yx'dt$$

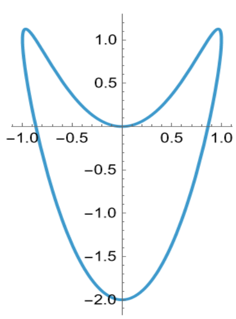
The area enclosed by the curve is

$$A = A_1 - A_2 = \int_a^c yx'dt - \int_b^c yx'dt = \int_a^b yx'dt$$

Example: find the area of the "eye" of the curve $x = t^2, y = t^3 - t$.
This curve is closed and simple when $-1 \leq t \leq 1$. The enclosed area is

$$A = \left|\int_a^b xy'dt\right| = \left|\int_{-1}^1 t^2(3t^2 - 1)dt\right| = \frac{8}{15}$$

Example: a simple heart has the equation $x = \sin t, y = \cos t - \cos 2t, 0 \leq t \leq 2\pi$.



The enclosed area is

$$A = \left|\int_0^{2\pi} yx'dt\right| = \left|\int_0^{2\pi} (\cos t - \cos 2t)\cos t\, dt\right|$$

Then use trigonometric identities to simplify the integrand.

Polar coordinates: (r, θ)
Conversion from polar coordinates (r, θ) to Cartesian coordinates (x, y):

$$x = r\cos\theta\,, y = r\sin\theta$$

Conversion from Cartesian coordinates (x, y) to polar coordinates (r, θ):

$$r = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$$

Note that θ is not unique. It is unique up to a multiple of 2π.

For graphing purposes, we will allow $r < 0$ so that r can be any function of θ. We will understand the point $(-r, \theta)$ as the point $(r, \theta + \pi)$. Two different pairs (r_1, θ_1) and (r_2, θ_2) are called *equivalent* if they represent the same point on the plane.

$$(r, \theta) \equiv (r, \theta + k2\pi)$$
$$(r, \theta) \equiv (-r, \theta + \pi + k2\pi)$$
$$(0, \theta) \equiv (0, \phi)$$

Practice converting Cartesian to polar coordinates and vice versa.
Practice finding equivalent polar coordinates.