Lecture 3

Wednesday, January 15, 2025 12:08 AM

Goal: Integrating trigonometric functions of the form $\sin^m x \cos^n x$ where m and n are integers.

If m or n is odd, we will use substitution $u = \cos x$ or $u = \sin x$ and the trigonometric identity $\sin^2 x + \cos^2 x = 1$.
If both m and n are even, we will use the double angle trigonometric identities

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos x \sin x = \frac{1}{2}\sin 2x$$

to lower the power or sine and cosine. You may need to do so more than once. Then use the sum identities:

$$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$
$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$
$$\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$$

Ex: $\int \sin^3 x \cos^2 x \, dx$
Use the substitution $u = \cos x$

Ex: $\int \sin^2 x \cos^4 x \, dx$
Note that

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1 + \cos 2x}{2}\right)^2 = \frac{(1 + \cos 2x)^2}{4} = \frac{1 + 2\cos 2x + \cos^2 2x}{4}$$
$$= \frac{1 + 2\cos 2x + \frac{1 + \cos 4x}{2}}{4}$$

Or you can note that

$$\sin^2 x \cos^4 x = (\sin x \cos x)^2 \cos^2 x = \left(\frac{1}{2}\sin 2x\right)^2 \cos^2 x = \frac{1}{4}\sin^2 2x \cos^2 x$$
$$= \frac{1}{4}\frac{1 - \cos 4x}{2}\frac{1 + \cos 2x}{2} = \frac{1}{8}(1 - \cos 4x)(1 + \cos 2x)$$
$$= \frac{1}{8}(1 - \cos 4x + \cos 2x - \cos 4x \cos 2x)$$
$$= \frac{1}{8}\left(1 - \cos 4x + \cos 2x - \frac{1}{2}(\cos 2x + \cos 6x)\right)$$
$$= \frac{1}{8}\left(1 - \cos 4x + \frac{1}{2}\cos 2x - \frac{1}{2}\cos 6x\right)$$

In terms of complex numbers:

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \quad and \quad \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

Then you can simplify $\sin^2 x \cos^4 x$.