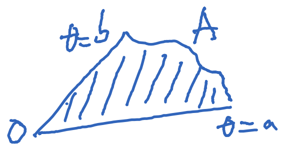
Lecture 32

Monday, March 10, 2025 12:18 PM

Area of a polar region between two rays $\theta = a$ and $\theta = b$.



Let $A(\theta)$ be the area swept by the polar curve when the angle ranges from a to θ.

Then $A(\theta + h) - A(\theta)$ is the area of the red region. If h is small, this region looks like a circular sector with radius $r(\theta)$ with angle h. Thus,

$$A(\theta + h) - A(\theta) \approx \frac{1}{2} r(\theta)^2 h$$

Then

$$\frac{A(\theta + h) - A(\theta)}{h} \approx \frac{1}{2} r(\theta)^2$$

Let $h \to 0$, we get $A'(\theta) = \frac{1}{2} r(\theta)^2$. Now note that $A(a) = 0$. By the Fundamental Theorem of Calculus,

$$A = \frac{1}{2} \int_a^b r(\theta)^2 d\theta$$

Example: What is the area enclosed by the heart curve $r = 1 + \sin\theta$?

Arclength of a polar curve between $\theta = a$ and $\theta = b$.

$$x = r\cos\theta = r(\theta)\cos\theta$$
$$y = r\sin\theta = r(\theta)\sin\theta$$

$$L = \int_a^b \sqrt{(x')^2 + (y')^2} d\theta = \int_a^b \sqrt{r^2 + (r')^2} d\theta$$

Example: What is the length of the heart curve above?

Example: Find the tangent line to a polar curve at the point $(r, \theta) = \left(1 + \sqrt{3}/2, \pi/3\right)$

The parametric equation of the curve is:

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta$$
$$y = r\sin\theta = (1 + \sin\theta)\sin\theta$$

At the point $(r, \theta) = \left(1 + \sqrt{3}/2, \pi/3\right)$, we have $x = (2 + \sqrt{3})/4$, $y = (2\sqrt{3} + 3)/4$.

The slope of the curve is

$$\frac{y'}{x'} = \frac{\cos\theta\sin\theta + (1 + \sin\theta)\cos\theta}{\cos^2\theta - (1 + \sin\theta)\sin\theta} = -1$$

Therefore, the equation of the tangent line is $y = \frac{2\sqrt{3}+3}{4} + (-1)\left(x - \frac{2+\sqrt{3}}{4}\right) = \frac{5+3\sqrt{3}}{4} - x$