Lecture 44

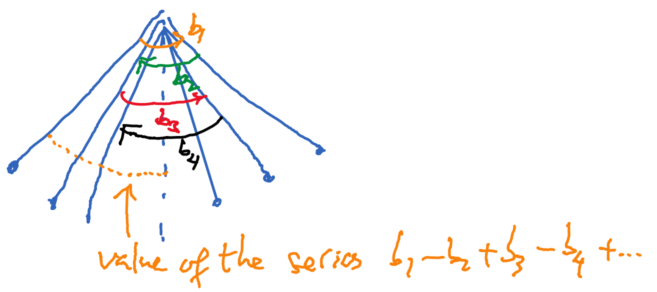
Friday, March 28, 2025 9:07 AM

Alternating Series Test (for convergence):
If b_n is a decreasing sequence and $\lim b_n = 0$ then the series converges.

Visualization: Imagine a pendulum swinging back and forth:

- At first, it swings wide — a big arc to the right (like adding b_1),
- Then swings back to the left, but not as far (subtracting b_2),
- Then right again, but less than before ($+b_3$),
- Then left again, smaller swing ($-b_4$),
- And so on...

Each swing is smaller than the last, and over time the pendulum comes to rest at a central point.



Exercises:

$$\sum (-1)^n \frac{n}{n+1}$$

$$\sum \frac{(-1)^n}{2^n}$$

$$\sum (-1)^n \frac{n^2}{2^n}$$

$$\sum (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$$

The third series in an interesting one: $b_n = n^2/2^n$.
$f(x) = x^2/2^x$. You can check that $f'(x) < 0$ for large x. Thus, f is a decreasing function on some interval $[M, \infty)$. This implies that starting at some index, b_n is decreasing.

$$\lim b_n = \lim_{x \to \infty} f(x) = \cdots = 0 \text{ (L'Hospital rule)}$$

Error estimate for alternating series:
A natural way to compute a series is to truncate the series, i.e. to approximate the series by a partial sum. Alternating series is interesting in that you can estimate the error quite easily.

Suppose that b_n is decreasing and $\lim b_n = 0$. The error by approximating the series by the m'th partial sum is

$$
\begin{aligned}
e_m = \sum_{n=1}^{\infty}(-1)^n b_n - \sum_{n=1}^{m}(-1)^n b_n &= (-1)^{m+1}b_{m+1} + (-1)^{m+2}b_{m+2} + \cdots \\
&= (-1)^{m+1}(b_{m+1} - b_{m+2} + b_{m+3} - b_{m+4} + b_{m+5} - b_{m+6} + \cdots)
\end{aligned}
$$

Take the absolute value of both sides:

$$
\begin{aligned}
|e_m| &= b_{m+1} - b_{m+2} + b_{m+3} - b_{m+4} + b_{m+5} - b_{m+6} + \cdots \\
&= b_{m+1} - (b_{m+2} - b_{m+3}) - (b_{m+4} - b_{m+5}) - \cdots \\
&\leq b_{m+1}
\end{aligned}
$$