Lecture 45

Monday, March 31, 2025 1:14 PM

Error estimate for alternating series:

$$\left|\sum_{n=1}^{\infty}(-1)^n b_n - \sum_{n=1}^{m}(-1)^n b_n\right| \le b_{m+1}$$

Example: find the value of $\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}$ correct to 3 decimal places.
This is an alternating series with $b_n = \frac{1}{n2^n}$. It converges according to the Alternating Series Test. If approximating this series by its m'th partial sum, the error estimate is

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n} - \sum_{n=1}^{m}\frac{(-1)^n}{n2^n}\right| \le \frac{1}{(m+1)2^{m+1}}$$

To get correct 3 decimal places, we want this error to be less than 10^{-3}. You can achieve this goal by choosing m such that

$$\frac{1}{(m+1)2^{m+1}} < 10^{-3} = \frac{1}{1000}$$

which is equivalent to $(m+1)2^{m+1} > 1000$. So, $m \ge 7$.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n} \approx \sum_{n=1}^{7}\frac{(-1)^n}{n2^n} = -\frac{1}{2}+\frac{1}{8}-\frac{1}{24}+\frac{1}{64}-\frac{1}{160}+\frac{1}{384}-\frac{1}{889} \approx \cdots$$

Two types of convergence:

- $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ converges.
- $\sum a_n$ is *conditionally convergent* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Examples:

$$\sum \frac{1}{n^2}$$

$$\sum \frac{(-1)^n}{n}$$

$$\sum \frac{\sin n}{n^2}$$

$$\sum \frac{1}{\ln n}$$

If $\sum a_n$ conditionally converges then a rearrangement of its term may change the value of the series. In fact, it can be rearranged to make it equal to any real number (Riemann rearrangement theorem).

Example:

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots$$

This series converges conditionally and the value is 0. However, the following rearrangement gives a different value:

$$\left(1+\frac{1}{2}-1\right)+\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{6}-\frac{1}{3}\right)+\left(\frac{1}{7}+\frac{1}{8}-\frac{1}{4}\right)+\cdots$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\frac{1}{2}+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\cdots>\frac{1}{2}$$