Lecture 5

Friday, January 17, 2025 1:56 AM

We have discussed how to find integrals of the form $\int \sin^m x \cos^n x \, dx$. There is another type of trigonometric integral that is mentioned in the textbook, namely

$$\int \tan^m x \sec^n x \, dx$$

If either m is odd or n is even, this type of integral is simpler compared to the former one. The strategy is to use substitution $u = \tan x$ (if n is even) or $u = \sec x$ (if m is odd). Keep in mind that:

$(\sec x)' = \tan x \sec x$

$(\tan x)' = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

Example: $\int tan\, x\, sec^3\, x\, dx$

Here, $n = 3$ is odd. Let $u = \sec x$. Then $du = u'dx = \tan x \sec x \, dx$.

$$\int \tan x \sec^3 x \, dx = \int \sec^2 x \tan x \sec x \, dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C$$

Example: $\int \tan x \sec^4 x \, dx$

Here, $n = 4$ is even. Let $u = \tan x$. Then $du = u'dx = \sec^2 x \, dx$.

$$\int \tan x \sec^4 x \, dx = \int \tan x \sec^2 x \sec^2 x \, dx = \int \tan x \,(1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u(1 + u^2)du = \int (u + u^3)du = \frac{u^2}{2} + \frac{u^4}{4} + C = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$$

<u>Goal for this lecture and next lecture</u>: deal with integrals that contain $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$. For example,

$$\int \frac{1}{\sqrt{1-x^2}} dx,\ \int \sqrt{1 - x^2} dx, \int \sqrt{3 - x^2} dx, \int x^2\sqrt{1 - x^2} dx, \int \sqrt{1 + x^2} dx, \int \frac{1}{\sqrt{2+3x^2}} dx$$

Note that we already knew at least one of them: $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

Tips:

For $\sqrt{a^2 - x^2}$, use the substitution $x = a \sin u$ or $x = a \cos u$.

For $\sqrt{x^2 - a^2}$, use the substitution $x = a \sec u$.

For $\sqrt{a^2 + x^2}$, use the substitution $x = a \tan u$.

Keep in mind the following identities:

$$\sqrt{1 - \cos^2 u} = |\sin u|$$
$$\sqrt{1 - \sin^2 u} = |\cos u|$$
$$\sqrt{\sec^2 u - 1} = |\tan u|$$
$$\sqrt{1 + \tan^2 u} = |\sec u|$$

Example: $\int_0^{\frac{1}{2}} \sqrt{1 - x^2} dx$

Example: $\int_0^1 x^2\sqrt{1 - x^2} dx$

Example: $\int_0^5 \sqrt{1 + x^2} dx$