Lecture 51

Thursday, April 10, 2025 9:09 AM

$$\frac{2x+1}{3x-5} = x \cdot \frac{2}{3x-5} + \frac{1}{3x-5}$$

What is the interval of convergence?

$$\frac{x^2}{x^2-3x+2} = x^2 \frac{1}{x^2-3x+2} = x^2\left(\frac{1}{x-3} - \frac{1}{x-2}\right)$$

Then represent $\frac{1}{x-3}$ and $\frac{1}{x-2}$ as power series. The interval of convergence of the combined series is the intersection of the intervals of convergence of those two power series.

If you can't relate a function $f(x)$ to the function $\frac{1}{1-x}$, how do you write it as a power series?

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Plugging $x = a$, you get $c_0 = a$.
Differentiating and then plugging $x = a$, you get $c_1 = f'(a)$.
Differentiating again and then plugging $x = a$, you get $c_2 = f''(a)/2$.
And so on. You get $c_n = \frac{f^{(n)}(a)}{n!}$. This means that there is only one power series centered at a that is equal to the function f. Namely,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We call this power series the Taylor series centered at a of the function f.

Example: for $f(x) = e^x$ and $a = 0$,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Theorem: Let $R > 0$ be the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$. Then for any $x \in (a-R, a+R)$,

$$\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int_a^x \sum_{n=0}^{\infty} c_n (t-a)^n \, dt = \sum_{n=0}^{\infty} \int_a^x c_n (t-a)^n dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

In other words, in the interval $(a-R, a+R)$, you can differentiate or integrate the power series by differentiating or integrating termwise. The "derivative series" and "integral series" have the same radius of convergence as the original power series.

Example:

$$\ln(x+1) = \int_0^x \frac{1}{t+1}\,dt = \int_0^x \sum_{n=0}^{\infty} (-t)^n\,dt = \sum_{n=0}^{\infty} \int_0^x (-t)^n dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$$

Example:

Compute the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$

The radius of convergence is $R = 1$. For $x \in (-1,1)$,

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

By Fundamental Theorem of Calculus,

$$f(x) = f(0) + \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x)$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = f\left(\frac{1}{2}\right) = -\ln\left(1-\frac{1}{2}\right) = \ln 2$$