Lecture 7

Wednesday, January 22, 2025 1:06 AM

Correction: for integrand of the form $\tan^m x \sec^n x$, we use the substitution

- $u = \tan x$ if n is even
- $u = \sec x$ if m is odd

The case m is even and n is odd is trickier. One will need to use a technique called *partial fraction decomposition*, which we will talk about today.

In the next 3 lectures, we will learn how to integrate a rational function $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials.

Consider the case $\deg Q > \deg P$:
Some integrals are quite easy to find using the substitution $u = Q(x)$.

Example:

$$\int \frac{2x-1}{x^2-x-2}\,dx$$

Let $u = x^2 - x - 2$. Then $du = (2x-1)dx$.

$$\int \frac{2x-1}{x^2-x-2}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2-x-2| + C$$

However, this substitution doesn't work for all integrals. For example,

$$\int \frac{3x-1}{x^2-x-2}\,dx$$

In that case, you should try to factor the denominator $Q(x) = x^2 - x - 2 = (x+1)(x-2)$. Then we decompose the rational function into simple fractions:

$$\frac{3x-1}{x^2-x-2} = \frac{3x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

where A and B are constants to be found. We find A and B by multiplying both sides by $(x+1)(x-2)$.

$$3x - 1 = A(x-2) + B(x+1)$$

With $x = 2$, we get $B = 5/3$. With $x = -1$, we get $A = 4/3$. Then

$$\int \frac{3x-1}{x^2-x-2}\,dx = \int \left(\frac{4/3}{x+1} + \frac{5/3}{x-2}\right)dx = \frac{4}{3}\ln|x+1| + \frac{5}{3}\ln|x-2| + C$$

In general, if $Q(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$ where $x_1, x_2, \ldots, x_n$ are distinct and $\deg P < \deg Q$, then we decompose the rational function $P(x)/Q(x)$ into simple fractions of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \cdots + \frac{A_3}{x-x_n}$$

This decomposition is called *partial fraction decomposition*. It simply means a sum of simplest possible rational functions. The coefficients $A_1, A_2, \ldots, A_n$ are constants and can be found by multiplying both sides by $Q(x)$ and then substituting $x = x_1, x_2, \ldots, x_n$ consecutively.