Lecture 8

Thursday, January 23, 2025 12:26 AM

Do Problem 3 on the last worksheet.

Recall the theorem:
If $\deg P < \deg Q$ and $Q(x) = (x - x_1)(x - x_2) \dots (x - x_n)$ where $x_1, x_2, \dots, x_n$ are distinct, then we decompose the rational function $P(x)/Q(x)$ into simple fractions of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \dots + \frac{A_3}{x - x_n}$$

What if $Q(x)$ has repeated roots? For example, $Q(x) = (x-1)^3(x-2)$ and $P(x) = x$.

$$\frac{x}{(x-1)^3(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{A_4}{x-2}$$

To find A_1, A_2, A_3, A_4, we multiply both sides by $(x-1)^3(x-2)$:

$$x = A_1(x-1)^2(x-2) + A_2(x-1)(x-2) + A_3(x-2) + A_4(x-1)^3$$

Plug $x = 1$: we get $1 = -A_3$, so $A_3 = -1$.
Plug $x = 2$: we get $2 = A_4$, so $A_4 = 2$.
Plug $x = 0$: we get $0 = -2A_1 + 2A_2 - 2A_3 - A_4 = -2A_1 + 2A_2 - 2(-1) - 2$, so $A_1 = A_2$.
Plug $x = 3$: we get $3 = 4A_1 + 2A_2 + A_3 + 8A_4 = 6A_1 + (-1) + 8$, so $A_1 = -2/3$.
Therefore, $A_1 = A_2 = -\frac{2}{3}, A_3 = -1, A_4 = 1$.

$$\int \frac{x}{(x-1)^3(x-2)} dx = \int \left(\frac{-2/3}{x-1} + \frac{-2/3}{(x-1)^2} + \frac{-1}{(x-1)^3} + \frac{1}{x-2} \right) dx$$

$$= -\frac{2}{3} \int \frac{dx}{x-1} - \frac{2}{3} \int \frac{dx}{(x-1)^2} - \int \frac{dx}{(x-1)^3} + \int \frac{dx}{x-2}$$

$$= -\frac{2}{3} \ln|x-1| + \frac{2}{3} \frac{1}{x-1} + \frac{1}{2} \frac{1}{(x-1)^2} + \ln|x-2| + C$$