Lecture 9

Friday, January 24, 2025 1:12 PM

What if $Q(x)$ cannot be factored? For example, $Q(x) = x^2 + 1$ and $P(x) = x - 1$.

$$\int \frac{x-1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) - \arctan x + C$$

Another example:

$$\int \frac{2x-1}{x^2+3}\,dx = \int \frac{2x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

You can find the first integral using the substitution $v = x^2 + 3$.
The second integral is a little more tricky. Observe that

$$\frac{1}{x^2+3} = \frac{1}{3\left(\frac{x^2}{3}+1\right)} = \frac{1}{3}\frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}$$

You can find the integral of this function using the substitution $w = x/\sqrt{3}$.

Another example:

$$\int \frac{x-1}{x^2+x+1}\,dx$$

Complete the square in the denominator: $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$.
Then use the substitution $u = x + \frac{1}{2}$.

$$\int \frac{x-1}{x^2+x+1}\,dx = \int \frac{u-3/2}{u^2+\frac{3}{4}}\,du = \int \frac{u}{u^2+\frac{3}{4}}\,du - \int \frac{3/2}{u^2+\frac{3}{4}}\,du$$

You can find the first integral using the substitution $v = u^2 + 3/4$. For the second integral, use the substitution $w = 2u/\sqrt{3}$.

Work on the problem on the worksheet.

What if $\deg P \geq \deg Q$? In this case, you need to do *long division* to write the fraction $P(x)/Q(x)$ in the form

$$\frac{P(x)}{Q(x)} = g(x) + \frac{r(x)}{Q(x)}$$

where $g(x)$ is the quotient and $r(x)$ is the remainder. Because $\deg r < \deg Q$, you can find the integral of $r(x)/Q(x)$ using the methods we discussed before.

Example:

$$\int \frac{x^3+x}{x-1}\,dx$$

Example:

$$\int \frac{x^4+x^2+1}{x^2(x+1)}\,dx$$