Homework 4

Consider the polynomial $f(x) = 2x^4 - 9x^3 + 18x - 4$.

1. Use Intermediate Value Theorem to show that f has a root in each of the intervals $(-2,-1)$, $(0,1)$, $(1,2)$, $(3,4)$. Call these roots by r_1, r_2, r_3, r_4 respectively.

2. Graph $f(x)$ on the interval $[-2,4]$. You may use GeoGebra and copy the graph onto paper by hand. For the best rendition, after you graph on GeoGebra, click on the gear icon ⚙ on the top right. Choose `Setting` and change the axis ratio `xAxis:yAxis` to $1:20$. Press Enter.

3. Just by looking at the graph, give a rough estimate for r_1, r_2, r_3, r_4.

4. Use the Bisection method to find r_1 on the interval $[a,b] = [-2,-1]$ with an allowable error of 0.01.

5. Use Newton-Raphson method to find r_3 with an allowable error of 0.0001. That is, you stop when $|x_{n+1} - x_n| < 0.0001$.

6. The Gradient Descent method is normally used to find a local minimum. However, it can also be used to find a root. Notice that $f(x) = 0$ if and only if $g(x) = 0$, where $g(x) = f(x)^2$. Since $g(x)$ is always nonnegative, 0 is its minimum value. Thus, the problem of finding a root of f can be translated to finding a number x at which g is minimum.

   (a) Graph the function $g(x)$ on the interval $[0,1]$. On GeoGebra, you might want to change the axis ratio `xAxis:yAxis` to $1:100$ for a good rendition.

   (b) Find r_2 using the Gradient Descent method with initial guess $x_0 = 0$ and learning rate $\alpha = 0.0015$. Stop when $|g'(x_n)| < 0.1$.

   (c) Do Part (b) with $\alpha = 0.0045$. What do you notice?

7. Just by looking at the graph of $f(x)$ (not of $g(x)$), give a rough estimate for the maximum and minimum values of f on $[-1,2]$ and also the values of x where they are attained.

8. Find the maximum value of f on the interval $[-1,2]$ using the Gradient Ascent method with initial guess $x_0 = 0$ and learning rate $\alpha = 0.03$. Stop when $|f'(x_n)| < 0.01$.

9. In Calculus I, you learned how to find the maximum value of f on $[-1,2]$ by first solving the equation $f'(x) = 0$ for all critical numbers on that interval. It is tricky to find the exact solutions because $f'(x)$ is a cubic polynomial. Instead, follow the steps below.

   (a) Use Newton-Raphson method to solve for all critical numbers of f on $[-1,2]$. Stop when $|x_{n+1} - x_n| < 0.0001$.

   (b) Use the critical numbers found in Part(a) to find the maximum value of f on $[-1,2]$.