Homework 8

1. Consider the initial-value problem

$$y' = y + x, \ y(0) = 1. \tag{1}$$

   (a) Find the exact solution $y = y(x)$.

   (b) Use Euler's method with step size 0.1 to estimate $y(0.3)$.

2. In Euler's method, you use the *forward difference approximation* $y'(x_n) \approx \frac{y(x_{n+1}) - y(x_n)}{h}$. In this exercise, you will improve the approximation by using the *centered difference approximation*:

$$y'(x_n) \approx \frac{y(x_{n+1}) - y(x_{n-1})}{2h} \tag{2}$$

   (a) With $y_n = y(x_n)$ and using the approximation (2), write the recursive formula for the differential equation $y' = y + x$. Your answer should be a formula of y_{n+1} in terms of y_n, y_{n-1}, and h (step size).

   (b) Notice that the recursive formula is completely determined if you know y_1 and y_0. Use the initial condition $y_0 = y(x_0) = y(0) = 1$. To find y_1, follow the steps below.

      (b1) Find the exact value of $y'(x_0)$ and $y''(x_0)$ directly from (1) without using the formula of the exact solution found in Part (a) of Problem 1.

      (b2) Then use the approximations

$$y'(x_0) \approx \frac{y_1 - y_{-1}}{2h}, \quad y''(x_0) \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

      to find y_1. You should get a formula involving h.

   (c) With step size 0.1, use the recursive formula obtained in Part (a) with y_0, y_1 found in Part (b) to estimate $y(0.3)$.

   (d) Compare the exact value of $y(0.3)$ with the approximated value obtained from the regular Euler's method (using forward difference approximation) and with the one obtained from the new method (using centered difference approximation). Which method gives a better approximation?

3. Consider the initial-valued problem $y'' + xy = x$, $y(0) = 0$, $y'(0) = 2$.

   (a) With $y_n = y(x_n)$ and using the *centered difference approximation for second derivative*

$$y''(x_n) \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2},$$

   write the recursive formula for the differential equation. Your answer should be a formula of y_{n+1} in terms of y_n, y_{n-1}, and h (step size).

   (b) Find y_1 by using the forward difference approximation for $y'(0)$.

   (c) Use the step size 0.1 to estimate $y(0.4)$.

4. Consider a rod of length 1 heated on one end. Think of this rod as the interval $[0, 1]$. The temperature at position $x \in [0, 1]$ at time $t \geq 0$ is denoted by $u(x, t)$, which satisfies the *heat equation*:

$$u_t - u_{xx} = 0$$

At all time, the left end is kept at constant temperature $u(0,t) = 1$ while the right end is kept at constant temperature $u(1,t) = 0$. Suppose the initial temperature on the rod is $u(x,0) = 1 - x^2$.

(a) The interval $[0,1]$ is discretized by $0 = x_0 < x_1 < ... < x_n = 1$ with $x_i - x_{i-1} = h$. The time is also discretized by $0 = t_0 < t_1 < t_2 < t_3 < ...$ with $t_j - t_{j-1} = \tau$. Denote $u_{i,j} = u(x_i, t_j)$. Approximate the derivatives u_t and u_{xx} at (x_i, t_j) using forward different in t and centered difference in x.

(b) Write a recursive formula of $u_{i,j}$ from the heat equation.

(c) Suppose $h = 0.25$ and $\tau = 0.02$. Find approximately the temperatures at the positions $x = 0.25, 0.5, 0.75$ at time $t = 0.04$.