Lecture 12

Friday, February 14, 2025 5:03 AM

Bisection method: bracket the root of $f(x) = 0$ with shorter and shorter intervals.
A search interval is an interval $[a, b]$ such that $f(a)f(b) < 0$. You calculate $c = \frac{a+b}{2}$. If $f(c)$ has the same sign as $f(a)$, you narrow the search interval to $[c, b]$. Otherwise, you narrow it to $[a, c]$. After n times of dividing, the search interval is of length $\frac{b-a}{2^n}$. The true solution lies on this interval. If you pick the midpoint of this interval, called x_*, then

$$|x_{exact} - x_*| < \frac{b-a}{2^{n+1}}$$

If you want the error to be lesson than some $\epsilon > 0$, you need to pick sufficiently large n such that

$$\frac{b-a}{2^{n+1}} < \epsilon$$

Example:
Find the intersection of the line $y = x$ and the curve $y = e^{-x}$.
The x-coordinate of the intersection is a root of $f(x) = x - e^{-x}$. You can check that $f'(x) > 0$, so f has a most one root. You can also check that $f(0) < 0$ and $f(1) > 0$. Thus, f has exactly one root, and it is between 0 and 1. Use the Bisection method to find the root with allowable error 0.01.

Newton-Raphson method (1669, 1699):
Starting at an initial guess x_0 , we approximate f by a linear function

$$f(x) \approx L(x) = f(x_0) + (x - x_0)f'(x_0)$$

And then solve the equation $L(x) = 0$. The result is called x_1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Then view x_1 as a new guess for root and repeat the process to get x_2, x_3,... In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This sequence, if converges, converges very fast to a solution to a root.

Example:
Do the previous example using Newton-Raphson method with initial guess $x_0 = 0$ with an allowable error of 0.0001. That is, you stop when $|x_{n+1} - x_n| < 0.0001$ and take x_{n+1} as the final answer.