Lecture 16

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At the point (x, y), the rate of change of f in the direction of *unit* vector u is

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

The reason why u must be a unit vector is because we are only interested in the direction of u. We normalize its length to be 1.

Question: for what direction u is the rate of change $D_u f(x,y)$ maximum?

$D_u f(x,y) = \nabla f(x,y) \cdot u = |\nabla f(x,y)||u| \cos\theta = |\nabla f(x,y)| \cos\theta \le |\nabla f(x,y)|$

The equality occurs when $\theta = 0$. That is, when u points to the direction of $\nabla f(x,y)$. In other words, the maximum rate of increase is in the direction

$$u = \frac{\nabla f}{|\nabla f|}$$

and the maximum rate is

$$\max D_u f(x,y) = |\nabla f(x,y)|.$$

The gradient vector is the direction of *steepest ascent*. The negative of the gradient vector is the direction of *steepest descent*. The direction tangent to the level set is the direction of zero rate of increase.

Work on some problems on the worksheet.

Find min/max of a multivariable function:

The familiar procedure for optimizing a single-variable function $f(x)$ on the interval $[a, b]$ is, first, to find all critical numbers of f, and then to compare the values of f at these critical numbers and at the endpoints of the interval. You determine the critical numbers because they are potentially places where f attains min/max.

The same procedure holds for multivariable functions. The equivalence of critical numbers are critical points (x, y) at which $\nabla f(x,y) = (0,0)$. Critical points are potentially places where f attains min/max. However, the "endpoints of the interval" will now have to be understood as the *boundary of the region* of (x, y) where you are to look for min/max. Finding min/max on the boundary is a more complex problem.

Example:

Find min/max of the function $f(x,y) = x^3 + xy^2 - 4x + 2y$ on the disk $x^2 + y^2 \le 4$.

$$\nabla f(x,y) = (f_x, f_y) = (3x^2 + y^2 - 4, 2xy + 2)$$

$\nabla f(x,y) = (0,0)$ if and only if $3x^2 + y^2 - 4 = 0$ and $2xy + 2 = 0$. From the second equation, you get $y = -1/x$. Substitute this y into the first equation:

$$3x^2 + \frac{1}{x^2} - 4 = 0$$

which can be rewritten as $3x^4 - 4x^2 + 1 = 0$. You can factor the polynomial:

$(x^2 - 1)(3x^2 - 1) = 0$

You get $x = \pm 1, \pm \frac{1}{\sqrt{3}}$. Therefore, you get 4 critical points:

$$(x,y) = (1,-1), (-1,1), \left(-\frac{1}{\sqrt{3}}, \sqrt{3}\right), \left(\frac{1}{\sqrt{3}}, -\sqrt{3}\right)$$

On the boundary of the disk $x^2 + y^2 \le 4$, we have $x^2 + y^2 = 4$. We will need to find min/max of $f(x,y)$ on the boundary, i.e. under the constraint $x^2 + y^2 = 4$.

You have $y^2 = 4 - x^2$. Then
$f(x,y) = x^3 + x(4 - x^2) - 4x + 2y = 2y$
On the circle of radius 2,

$\max 2y = 4$ (attained when $x = 0, y = 2$)

$\min 2y = -4$ (attained when $x = 0, y = -2$)

Therefore,

$$\max f = \max\left\{f(-1,1), f(1,-1), f\left(-\frac{1}{\sqrt{3}}, \sqrt{3}\right), f\left(\frac{1}{\sqrt{3}}, -\sqrt{3}\right), f(0,2)\right\} = \cdots$$

$$\min f = \min\left\{f(-1,1), f(1,-1), f\left(-\frac{1}{\sqrt{3}}, \sqrt{3}\right), f\left(\frac{1}{\sqrt{3}}, -\sqrt{3}\right), f(0,-2)\right\} = \cdots$$