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Find min/max of a multivariable function:

The familiar procedure for optimizing a single-variable function f(x) on the interval [a, b] is, first, to find all critical numbers of f, and then to compare the values of f at these critical numbers and at the endpoints of the interval. You determine the critical numbers because they are potentially places where f attains min/max.

The same procedure holds for multivariable functions. The equivalence of critical numbers are critical points (x, y) at which $\nabla f(x, y) = (0, 0)$. Critical points are potentially places where f attains min/max. However, the "endpoints of the interval" will now have to be understood as the *boundary* of the region of (x, y) where you are to look for min/max. Finding min/max on the boundary is a more complex problem.

Example:

Find min/max of the function $f(x, y) = x^3 + xy^2 - 4x + 2y$ on the disk $x^2 + y^2 \le 4$. $\nabla f(x, y) = (f_x, f_y) = (3x^2 + y^2 - 4, 2xy + 2)$

 $\nabla f(x, y) = (0, 0)$ if and only if $3x^2 + y^2 - 4 = 0$ and 2xy + 2 = 0. From the second equation, you get y = -1/x. Substitute this *y* into the first equation:

$$3x^2 + \frac{1}{x^2} - 4 = 0$$

which can be rewritten as $3x^4 - 4x^2 + 1 = 0$. You can factor the polynomial: $(x^2 - 1)(3x^2 - 1) = 0$

You get $x = \pm 1, \pm \frac{1}{\sqrt{3}}$. Therefore, you get 4 critical points:

$$(x, y) = (1, -1), (-1, 1), \left(-\frac{1}{\sqrt{3}}, \sqrt{3}\right), \left(\frac{1}{\sqrt{3}}, -\sqrt{3}\right)$$

On the boundary of the disk $x^2 + y^2 \le 4$, we have $x^2 + y^2 = 4$. We will need to find min/max of f(x, y) on the boundary, i.e. under the constraint $x^2 + y^2 = 4$.

You have $y^2 = 4 - x^2$. Then

 $f(x, y) = x^3 + x(4 - x^2) - 4x + 2y = 2y$ On the circle of radius 2,

 $m_{\rm ext} 2 x = 4$ (attained whe

max 2y = 4 (attained when x = 0, y = 2)

min 2y = -4 (attained when x = 0, y = -2) Therefore,

$$\max f = \max\left\{f(-1,1), f(1,-1), f\left(-\frac{1}{\sqrt{3}}, \sqrt{3}\right), f\left(\frac{1}{\sqrt{3}}, -\sqrt{3}\right), f(0,2)\right\} = \cdots$$
$$\min f = \min\left\{f(-1,1), f(1,-1), f\left(-\frac{1}{\sqrt{3}}, \sqrt{3}\right), f\left(\frac{1}{\sqrt{3}}, -\sqrt{3}\right), f(0,-2)\right\} = \cdots$$

Example:

Find min/max of the function $f(x, y) = -x + xy + y^2$ on the triangle with vertices at (-3,0), (-2,2), (0,0).