Lecture 20

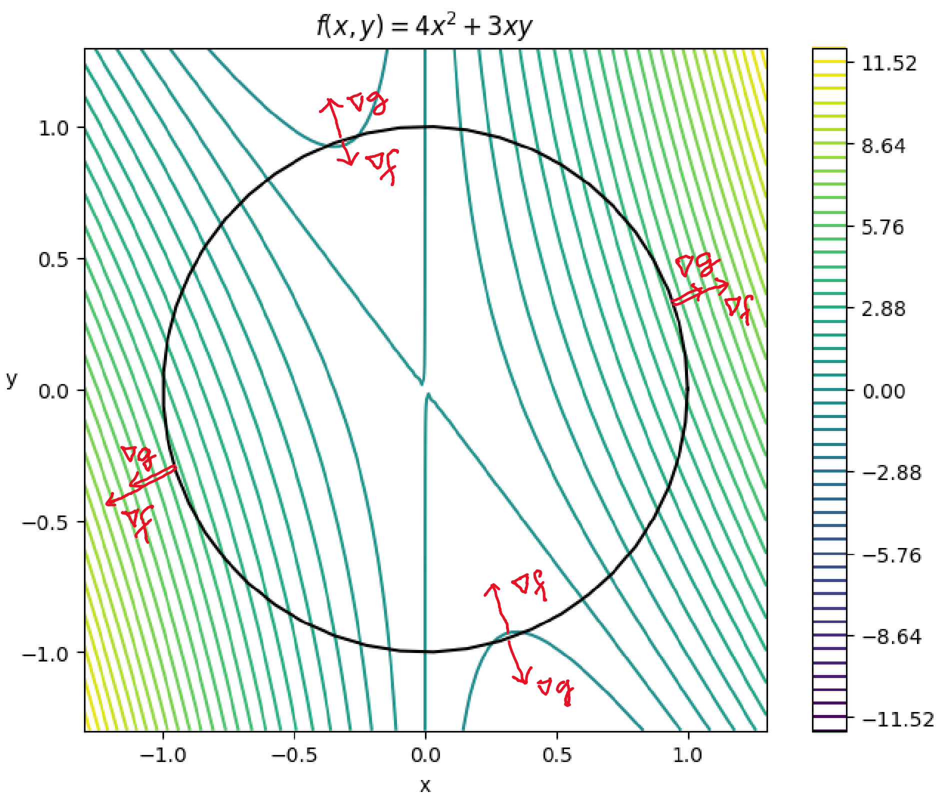
Wednesday, March 12, 2025 1:22 AM

Theorem: If f is continuous on a closed and bounded region D then it has minimum and maximum values on D.

Finding min/max of a function $f(x,y)$ on the boundary of a region D is equivalent to finding min/max of f under a *constraint*. This constraint is of equality type.

For example: find min/max of $f(x,y) = 4x^2 + 3xy$ on the boundary of the unit circle.

The boundary unit circle is closed and bounded, so f admits a minimum value and a maximum value there. The problem is equivalent to finding min/max of f under the constraint $x^2 + y^2 = 1$. Last time, we consider an example where the function is significantly reduced under such a constraint. Unfortunately, this scenario is not always the case, such as in the present example. You can write $y = \pm\sqrt{1 - x^2}$ and plug it in the formula of f and then reduce the problem to one-variable optimization. This is quite inconvenient, especially because you have to consider both possible formula for y.



Let $f(x,y) = 4x^2 + 3xy$ and $g(x,y) = x^2 + y^2$. We want to minimize/maximize f under the constraint $g(x,y) = 0$. The min/max of f is attained where the level curves of f just touches the 0-level curve of g. At this touching point, the level curve of f is tangent to the level curve of g. Therefore, the gradient of f (which is perpendicular to the level curve of f) must be parallel to the gradient of g which is perpendicular to the 0-level curve of g):

$$\nabla f = \lambda \nabla g$$

where λ is a number called Lagrange multiplier. This equation together with the equation $g = 0$ constitutes a system of 3 equations and 3 unknowns x, y, λ.

$f_x = 8x + 3y, g_x = 2x$
$f_y = 3x,\ g_y = 2y$

The system to solve for x, y, λ is

$$8x + 3y = \lambda 2x$$
$$3x = \lambda 2y$$
$$x^2 + y^2 = 1$$

From here, you can get $\lambda = -1/2$ and $\lambda = 9/2$. Then you substitute it back to the equation to solve for x, y. You will get 4 points (x, y).

Then you compare the values of f at these 4 points. The largest of them is the maximum of f. The smallest of them is the minimum of f.