Lecture 21

Friday, March 14, 2025 2:22 PM

Theorem: If f is continuous on a closed and bounded region then it has minimum and maximum values on that region.

Recall the procedure to find min/max of a function f on a region:

- Step 1: find all critical points of f *inside* that region
- Step 2: find min/max of f on the boundary
- Step 3: compare and conclude

Lagrange multiplier method is specialized for Step 2, although it is not the only method that is available.

Work on the example on the worksheet:
Find the point on the ellipse $x^2 + 4y^2 = 1$ that is closest to the point $(2,3)$.

The objective function is: $f(x,y) = (x-2)^2 + (y-3)^2$.
We want to minimize this function subject to the constraint $g(x,y) = 0$ where

$$g(x,y) = x^2 + 4y^2 - 1.$$

Computation of partial derivatives:

$$f_x = 2(x-2),\ f_y = 2(y-3)$$
$$g_x = 2x,\ g_y = 8y$$

The system to solve for x, y, λ:

$$2(x-2) = 2\lambda x$$
$$2(y-3) = 8\lambda y$$
$$x^2 + 4y^2 = 1$$

From the first equation, we get $x = \frac{2}{1-\lambda}$.
From the second equation, we get $y = \frac{3}{1-4\lambda}$.
Substituting these into the last equation, we get

$$\frac{4}{(1-\lambda)^2} + \frac{36}{(1-4\lambda)^2} = 1$$

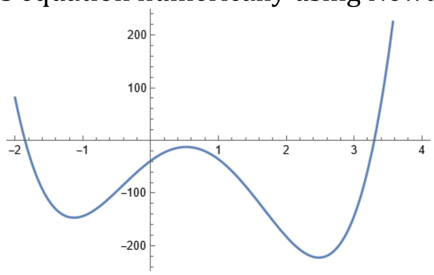
which is equivalent to

$$4(1-4\lambda)^2 + 36(1-\lambda)^2 = (1-\lambda)^2(1-4\lambda)^2$$

which is equivalent to

$$16\lambda^4 - 40\lambda^3 - 67\lambda^2 + 94\lambda - 39 = 0.$$

You can solve this equation numerically using Newton's method.



$$f(\lambda) = 16\lambda^4 - 40\lambda^3 - 67\lambda^2 + 94\lambda - 39$$
$$\lambda_{n+1} = \lambda_n - \frac{f(\lambda_n)}{f'(\lambda_n)}$$

Note: you can use Newton's method to find roots of the function

$$g(\lambda) = \frac{4}{(1-\lambda)^2} + \frac{36}{(1-4\lambda)^2} - 1$$