# Lecture 26

Friday, March 28, 2025 2:13 AM

Euler's method is an example of a more general method for numerically solving differential equations called *finite difference method*. The idea is that you approximate the derivative by the difference quotient.

**Example:**

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}$$

$$y'(x) \approx \frac{y(x) - y(x-h)}{h}$$

$$y''(x) \approx \frac{y'(x+h) - y'(x)}{h} \approx \frac{\frac{y(x+h)-y(x)}{h} - \frac{y(x)-y(x-h)}{h}}{h}$$

$$= \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

To see how good these approximations are when $h$ gets smaller and smaller, we use the **Taylor expansion**:

$$y(x+h) = y(x) + y'(x)h + \frac{y''(x)}{2!}h^2 + \frac{y'''(x)}{3!}h^3 + O(h^4)$$

By replacing $h$ with $-h$, we see that

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h)$$

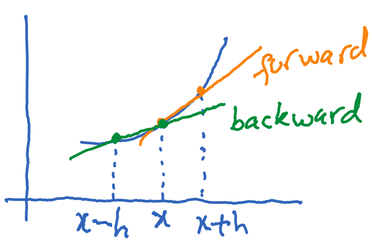
$$y'(x) = \frac{y(x) - y(x-h)}{h} + O(h)$$

$$y'(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2)$$

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2)$$

The $O(h)$ notation means "of order $h$", which means a quantity whose absolute value is less than or equal to $ch$ for some constant $c$.

The first equation is called *forward difference approximation* for derivative.
The second equation is called *backward difference approximation* for derivative.
The third equation is called *centered difference approximation* for derivative.
The four equation is called *centered difference approximation* for second derivative.




**Example:** $y'' + xy' + y = 1, y(-1) = 2, y'(-1) = 1$

$y(x_n) = y_n$

Let us use the centered difference approximation for $y''$:

$$y''(x_n) \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

This approximation is of order $h^2$ as $h$ gets small.

If you choose the approximation $y'(x_n) \approx \frac{y_n - y_{n-1}}{h}$ then the differential equation is approximated by

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + x_n \frac{y_n - y_{n-1}}{h} + y_n = 1$$

This yields

$$y_{n+1} = (2 - h^2 - hx_n)y_n + (hx_n - 1)y_{n-1} + h^2 \qquad (1)$$

Because the approximation of derivative is of order $h$ while the approximation of second derivative is of order $h^2$, the combined approximation is of order $h$ (the worse of the two aproximations).

On the other hand, if you choose the approximation $y'(x_n) \approx \frac{y_{n+1} - y_{n-1}}{2h}$, which is of order $h^2$ then the combined approximation is also of order $h^2$.

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + x_n \frac{y_{n+1} - y_{n-1}}{2h} + y_n = 1$$

This yields

$$y_{n+1} = \frac{(2 - h^2)y_n + \left(\frac{hx_n}{2} - 1\right)y_{n-1} + h^2}{1 + \frac{hx_n}{2}} \qquad (2)$$

To compute $y_{n+1}$, you need to know $y_n$ and $y_{n-1}$. You know $y_0 = 2$. How to find $y_1$? Once you have $y_1$, you will be able to find $y_2, y_3, y_4, \ldots$ using the above recursive formula ((1) or (2), depending on how you approximate the first derivative).

If you choose the forward difference approximation for $y'$ then $y_1$ is easy to find:

$$y'(x_0) \approx \frac{y_1 - y_0}{h}$$

$$y_1 \approx y_0 + hy'(x_0) = 2 + h$$

If you choose the centered difference approximation for $y'$ then

$$y'(x_0) \approx \frac{y_1 - y_{-1}}{2h}$$

which yields $y_1 = y_{-1} + 2h$. Applying (2) for $n = 0$, you get $y_1$ in terms of $y_0$ (known) and $y_{-1}$. From there, you can solve for $y_1$ (as well as $y_{-1}$).

**Exercise:** with step size $h = 0.1$, approximate $y(-0.7)$ using the recursive formula (1). Compare it with the true solution $y(x) = 1 + e^{(1-x^2)/2}$.

**Python code:** (using recursive formula (1))

```
from numpy import *
from matplotlib.pyplot import *
# solve y''+xy'+y=1 with initial condition y(x0)=a and y'(x0)=b
x0 = -1
a = 2
b = 1
# with step size h
h = 0.1
# the number of steps
N = 20
# Array x = [x0,x1,...,xN]
x = linspace(x0,x0+N*h,N+1)
# Array y = [y0,y1,...,yN]
y = zeros(N+1)
y[0] = a
y[1] = a+b*h
for n in range(1,N):
    y[n+1] = (2-h**2-h*x[n])*y[n] + (h*x[n]-1)*y[n-1] + h**2
plot(x,y)
plot(x,1+e**((1-x**2)/2))
show()
```